ON RECENT WORK OF TIAN AND KOLLÁR–TIAN

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Abstract. This is a survey on recent work of Tian [24] and Kollár–Tian [14], based on the mini-course that I gave at the Lodha Mathematical Sciences Institute on January 2026.

Contents

1. Introduction

Let $\mathbb{F}_q$ be a finite field, let ℓ be a prime invertible in $\mathbb{F}_q$, and let $C/\mathbb{F}_q$ be a smooth projective geometrically connected curve. Let $Y/\mathbb{F}_q(C)$ be a smooth projective geometrically connected variety, and for each closed point $v \in C$, let $Y_v := Y \times_{\mathbb{F}_q(C)} \mathbb{F}_q(C)_v$. Restricting Brauer classes on Y to each closed point of Y_v and composing with the norm map yield a pairing:

$$CH_0(Y_v) \times \mathrm{Br}(Y)\{\ell\} \to \mathrm{Br}(\mathbb{F}_q(C)_v)\{\ell\} \xrightarrow[\sim]{\mathrm{inv}} \mathbb{Q}_\ell/\mathbb{Z}_\ell.$$

The Albert–Brauer–Hasse–Noether theorem then yields a complex

$$\varprojlim_n CH_0(Y)/\ell^n \to \prod_{v\in C} \varprojlim_n CH_0(Y_v)/\ell^n \to \mathrm{Hom}(\mathrm{Br}(Y)\{\ell\}, \mathbb{Q}_\ell/\mathbb{Z}_\ell). \tag{1.1}$$

Conjecture 1.1 (Colliot-Thélène)**.** *The complex* (1.1) *is exact.*

The purpose of these notes is to discuss the following recent breakthrough towards Conjecture 1.1 due to Tian.

Theorem 1.2 (Tian, [24, Theorem 1.4])**.** *If $Y/\mathbb{F}_q(C)$ is a geometrically rational surface, then the complex* (1.1) *is exact.*

As a consequence of Theorem 1.2, Tian shows that if $\mathbb{F}_q$ is of odd characteristic, then the Brauer–Manin obstruction for the Hasse principle for del Pezzo surfaces of degree 4 over $\mathbb{F}_q(C)$ is the only one; see [24, Theorem 1.5].

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For $Y/\mathbb{F}_q(C)$ as in Theorem 1.2, the resolution of singularities shows that there exists a smooth projective geometrically connected 3-fold $X/\mathbb{F}_q$ with a dominant morphism $X \to C$ such that $X_\eta = Y$. By a result of Saito [18] together with a result of Esnault–Wittenberg [9], Theorem 1.2 is reduced to the integral Tate conjecture for 1-cycles on X. More precisely, Theorem 1.2 follows from the following result.

Theorem 1.3 (Tian, [24, Theorem 1.7])**.** *Let $X/\mathbb{F}_q$ be a smooth projective geometrically connected 3-fold with a dominant morphism $X \to C$ whose generic fiber $X_\eta/\mathbb{F}_q(C)$ is a smooth geometrically rational surface. Then the cycle map*

$$\mathrm{cl}^2\colon CH^2(X)\otimes\mathbb{Z}_\ell \to H^4(X,\mathbb{Z}_\ell(2))$$

is surjective.

In the special case when $Y/\mathbb{F}_q(C)$ has a conic bundle structure over $\mathbb{P}^1_{\mathbb{F}_q(C)}$, Theorem 1.2 and Theorem 1.3 were previously known due to Parimala–Suresh [16].

Towards the proof of Theorem 1.3, let $X/\mathbb{F}_q$ be a smooth projective geometrically connected variety. The Hochschild–Serre spectral sequence induces the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & CH^i(X)_{\mathrm{hom}}\otimes\mathbb{Z}_\ell & \longrightarrow & CH^i(X)\otimes\mathbb{Z}_\ell & & & & \\ & & \downarrow{\scriptstyle \mathrm{cl}^i_{\mathrm{AJ}}} & & \downarrow{\scriptstyle \mathrm{cl}^i} & \searrow^{\overline{\mathrm{cl}}^i} & & & \\ 0 & \longrightarrow & H^1(\mathbb{F}_q, H^{2i-1}(X_{\overline{\mathbb{F}}_q},\mathbb{Z}_\ell(i))) & \longrightarrow & H^{2i}(X,\mathbb{Z}_\ell(i)) & \longrightarrow & H^{2i}(X_{\overline{\mathbb{F}}_q},\mathbb{Z}_\ell(i))^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)} & \longrightarrow & 0, \end{array} \tag{1.2}$$

where the square is a pull-back. Hence cl^i is surjective if and only if $\overline{\mathrm{cl}}^i$ and $\mathrm{cl}^i_{\mathrm{AJ}}$ are both surjective.

For $X/\mathbb{F}_q$ as in Theorem 1.3, the surjectivity of $\overline{\mathrm{cl}}^2$ (geometric part) and the surjectivity of $\mathrm{cl}^2_{\mathrm{AJ}}$ (arithmetic part) will be addressed separately, but the proofs are based on the same geometric idea. To elaborate, we start with the following toy example.

Example 1.4. Let $X \subset \mathbb{P}^4_{\mathbb{F}_q}$ be a smooth cubic 3-fold. We aim to show that

$$\mathrm{cl}^2\colon CH^2(X)\otimes\mathbb{Z}_\ell \to H^4(X,\mathbb{Z}_\ell(2))$$

is surjective. A key player here for both the geometric and arithmetic parts is the Fano scheme $F(X)/\mathbb{F}_q$ of lines on X, which is a smooth projective geometrically connected surface. Let $U(X) \to F(X)$ be the universal line.

Geometric part: We first observe that, by weak Lefschetz, $H^4(X_{\overline{\mathbb{F}}_q},\mathbb{Z}_\ell(2)) = \mathbb{Z}_\ell$ is generated by the class $[L]$ of lines on $X_{\overline{\mathbb{F}}_q}$. Moreover, since $F(X)$ is geometrically irreducible, by Lang–Weil, $F(X)$ has a 0-cycle α of degree 1. Then $U(X)_*\alpha \in CH^2(X)$ is a 1-cycle of degree 1, and it represents $[L] \in H^4(X_{\overline{\mathbb{F}}_q},\mathbb{Z}_\ell(2))$. This shows that $\overline{\mathrm{cl}}^2$ is surjective.

Arithmetic part: By a result of Clemens–Griffiths [5, Theorem 11.19], we have

$$U(X)_*\colon H_1(F(X)_{\overline{\mathbb{F}}_q},\mathbb{Z}_\ell) \xrightarrow{\sim} H^3(X_{\overline{\mathbb{F}}_q},\mathbb{Z}_\ell(2)). \tag{1.3}$$

This induces the following diagram:

$$\begin{array}{ccc} CH_0(F(X))_{\mathrm{hom}}\otimes\mathbb{Z}_\ell & \xrightarrow[\sim]{\mathrm{cl}_{\mathrm{AJ},0,F(X)}} & H^1(\mathbb{F}_q, H_1(F(X)_{\overline{\mathbb{F}}_q},\mathbb{Z}_\ell)) \\ \downarrow{\scriptstyle U(X)_*} & & \wr\downarrow{\scriptstyle U(X)_*} \\ CH^2(X)_{\mathrm{hom}}\otimes\mathbb{Z}_\ell & \xrightarrow{\mathrm{cl}^2_{\mathrm{AJ},X}} & H^1(\mathbb{F}_q, H^3(X_{\overline{\mathbb{F}}_q},\mathbb{Z}_\ell(2))). \end{array}$$

Here $\mathrm{cl}_{\mathrm{AJ},0,F(X)}$ is an isomorphism by a theorem of Colliot-Thélène–Sansuc–Soulé [6, Théorème 5]: the Abel–Jacobi map for 0-cycles on every smooth projective geometrically connected $\mathbb{F}_q$-variety is an isomorphism. It follows that $\mathrm{cl}^2_{\mathrm{AJ},X}$ is surjective, and we conclude that cl^2_X is surjective.

In Example 1.4, for a smooth cubic 3-fold $X \subset \mathbb{P}^4_{\mathbb{F}_q}$, the following properties of $F(X)$ were crucially used:

(1) The *geometric irreducibility* of $F(X)$ is used to descend the class $[L]$ on $X_{\overline{\mathbb{F}}_q}$ to X in the geometric part.
(2) The *smoothness, projectiveness, and geometric connectedness* of $F(X)$ together with (1.3) are used to deduce the arithmetic part from a theorem of Colliot-Thélène–Sansuc–Soulé.

For $X/\mathbb{F}_q$ as in Theorem 1.3, we will first show that the cycle map $CH^2(X_{\overline{\mathbb{F}}_q}) \otimes \mathbb{Z}_\ell \to H^4(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell(2))$ is surjective. (This step may be deduced relatively easily from known results.) Then, to address respectively the geometric and arithmetic parts, we will construct alternatives in our situation to the Fano scheme of a cubic 3-fold. Namely, we will aim to show:

(1) For each class $\alpha \in H^4(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell(2))^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)}$, there exist a 1-cycle β on X, a geometrically irreducible variety $B_\alpha/\mathbb{F}_q$, a closed subvariety $U_\alpha \subset B_\alpha \times X$ with $U_\alpha \to B_\alpha$ dominant of relative dimension 1, and $b \in B^{sm}(\overline{\mathbb{F}}_q)$ such that $\alpha + \beta_{\overline{\mathbb{F}}_q} = U_b$.
(2) There exist a smooth projective geometrically connected variety $B/\mathbb{F}_q$, closed subvarieties $U_j \subset B \times X$ with $U_j \to B$ dominant of relative dimension 1, and integers n_j such that $(\sum_j n_j U_j)_* \colon H_1(B_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell) \twoheadrightarrow H^3(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell(2))$.

In general, the families $U_\alpha \to B_\alpha$ and $U_j \to B$ may be all distinct. The constructions of these families will be possible by the following key idea developed by Kollár–Tian [14] and Tian [24]:

> *For $X/\mathbb{F}_q$ as in Theorem 1.3, every family $U_0 \to B_0$ of curves on X with B_0 geometrically connected is a subfamily of another family $U_1 \to B_1$ of curves on X with B_1 smooth and geometrically connected, up to minor modifications of $U_0 \to B_0$.*

In fact, the above idea may be carried out for a larger class of varieties, i.e., for varieties that are *src in codimension* 1, which will be introduced in the next section.

Many steps of the proof of Theorem 1.3 will then be established, in most generalized ways possible, for 1-cycles on varieties that are src in codimension 1. The main theorems towards the geometric and arithmetic parts of Theorem 1.3 will respectively be: Theorem 3.2 on descending algebraic equivalence for 1-cycles, and Theorem 4.3 on coniveau and strong coniveau filtrations. The latter will follow from Theorem 6.2 and Theorem 7.2, which respectively analyze the image of cylinder homomorphisms on Lawson homology over the complex numbers and on higher Chow groups over algebraically closed fields. As for the aforementioned key idea on modifying families of curves, see Theorem 2.6, Theorem 3.4, and Theorem 5.2 for more precise statements.

Here is an outline of the survey. In Section 2, we introduce the notion of varieties that are src in codimension 1, discuss comb constructions for spaces of curves, and prove Theorem 2.6. In Section 3, we prove Theorem 3.2 on descending algebraic equivalence for 1-cycles, as well as Theorem 3.4 on deformation equivalences of curves. We then prove the geometric part of Theorem 1.3. In Section 4, we recall the notions of coniveau and strong coniveau filtrations, and state Theorem 4.3, whose proof is postponed until Section 6 and Section 7. We then relate the two coniveau filtrations to the surjectivity of the ℓ-adic Walker Abel–Jacobi maps by Proposition 4.4, and assuming Theorem 4.3, prove the arithmetic part of Theorem 1.3. In Section 5, we introduce the notion of equi-dimensional families of cycles, and state Theorem 5.2, which is a key technical result towards the proof of Theorem 4.3. In Section 6, we introduce Lawson homology for complex projective varieties, and prove Theorem 6.2 on the image of cylinder homomorphisms on Lawson homology, which yields the complex case of Theorem 4.3. In Section 7, we introduce higher Chow groups, and prove Theorem 7.2 on cylinder homomorphisms on higher Chow groups. We then deduce from Theorem 7.2 the general case of Theorem 4.3.

Many of the proofs given in this survey are only sketches, and I recommend the reader to consult the original papers [14, 24] for the details. I am responsible for all inaccuracies found in this survey.

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Conventions. We denote by k a base field and by ℓ a prime invertible in k. A k-variety means a separated scheme of finite type over k. When k is algebraically closed field, we simply call a k-variety a variety. For a k-variety X, we denote by X^{sm} the smooth locus of X. We denote by $x \in X$ a closed point x of X, unless otherwise specified. A node (resp. 3-node) of a k-curve is a singularity such that the completion of the local ring is isomorphic to $k[[x,y]]/(xy)$ (resp. $k[[x,y,z]]/(xy,yz,xz)$). We say that a k-curve is nodal (resp. 3-nodal) if it either is smooth or only admits nodes as singularities (resp. either is smooth or only admits nodes or/and 3-nodes as singularities). Following [14], for k-curves C, D, we denote by $C \uplus D$ the union of C and D such that all intersection points of C and D are nodes.

2. Varieties that are src in codimension 1 and comb constructions for spaces of curves

2.1. Varieties that are src in codimension 1.

Let k be an algebraically closed field.

Definition 2.1. Let X be a smooth projective connected variety.

- We say that X is *separably rationally connected*, or *src* in short, if there exists $f\colon \mathbb{P}^1 \to X$ such that f^*T_X is ample, or equivalently,
$$f^*T_X = \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(a_i),$$
where $a_i > 0$ for all i. Such f is called *very free*.
- We say that X is *separably rationally connected in codimension* 1, or *src in codimension* 1 in short, if there exists $f\colon \mathbb{P}^1 \to X$ such that
$$f^*T_X = \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(a_i),$$
where $a_i > 0$ for all i possibly except one $a_i = 0$. Such f is called *almost very free*.

If X is src, then there exists an irreducible component of $\mathrm{Mor}(\mathbb{P}^1, X)$ for which the two point evaluation map is dominant and separable. This is where the term "separably rationally connected" comes from.

It is helpful to know that being src (resp. being src in codimension 1) is a birational property and it is also an open condition in smooth projective families.

Example 2.2. Here are basic examples.

- $\mathbb{P}^n$ is src: leting $f\colon \mathbb{P}^1 \to \mathbb{P}^n$ be a line, we have
$$f^*T_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-1} \oplus \mathcal{O}_{\mathbb{P}^1}(2),$$
thus f is very free. More generally, rational varieties are src.
- A src fibration $X \to C$ over a curve is src in codimension 1: letting $f\colon \mathbb{P}^1 \to X_c$ be very free for some $c \in C$ and letting $g\colon \mathbb{P}^1 \xrightarrow{f} X_c \hookrightarrow X$ be the composition, we have
$$g^*T_X = f^*T_{X_c} \oplus \mathcal{O}_{\mathbb{P}^1},$$
thus g is almost very free.
- $X \times Y$, where X is src in codimension 1 and Y is src, is src in codimension 1: letting $f\colon \mathbb{P}^1 \to X$ be almost very free, $g\colon \mathbb{P}^1 \to Y$ be very free, and $(f,g)\colon \mathbb{P}^1 \to X \times Y$ be the induced map, we have
$$(f,g)^*T_{X\times Y} = f^*T_X \oplus g^*T_Y,$$
thus (f,g) is almost very free.

In characteristic 0, X is src in codimension 1 if and only if X is either src or birational to a src fibration over a curve of positive genus, but in positive characteristic, there are other types of varieties that are src in codimension 1; see [24, Lemma 2.2].

2.2. **Comb constructions for spaces of curves.** Let k be an algebraically closed field.

Definition 2.3. A *comb* with handle C_0 and teeth $T_1, \cdots, T_r$ is the union of nodal connected curves $C_0, T_1, \cdots, T_r$ such that $T_1, \cdots, T_r$ are disjoint and each T_i meets C_0 transversally at a single smooth point $p_i \in C_0$. A 3-*nodal comb* with handle C_0 and teeth $T_1, \cdots, T_r$ is the union of nodal connected curves $C_0, T_1, \cdots, T_r$ such that $T_1, \cdots, T_r$ are disjoint and each T_i meets C_0 either transversally at a single smooth point $p_i \in C_0$ or in a 3-nodal way at a single nodal point $p_i \in C_0$.

In this survey, a comb construction will mean the process of attaching to a nodal connected curve C_0 nodal connected curves $T_1, \cdots T_r$ to form a comb $C = C_0 \cup T_1 \cup \cdots \cup T_r$. A key observation on comb constructions is that if $T_1, \cdots, T_r$ are positive in an appropriate sense, the comb C has a better deformation property than C_0. More precisely, we have the following.

Lemma 2.4 (Graber–Harris–Starr, [11])**.** *Let X be a smooth projective connected variety. Let C be an embedded comb in X with handle C_0, teeth $T_1, \cdots, T_r$, and intersection points $C_0 \cap T_i = \{p_i\}$. If the number of teeth attached to each irreducible component of C_0 is sufficiently large, if p_i are general, and moreover if*

(1) each T_i is in a general normal direction to C_0 at p_i, and
(2) $H^1(T_i, N_{T_i}(-p_i)) = 0$,

then $H^1(C, N_C(-p)) = 0$ for all $p \in C_0^{sm}$. In particular, $H^1(C, N_C) = 0$ and $[C] \in \mathrm{Hilb}(X)^{sm}$.

Lemma 2.4 shows that the comb construction translates $[C_0] \in \mathrm{Hilb}(X)$ to $[C] \in \mathrm{Hilb}(X)^{sm}$. Our goal is to perform comb constructions for spaces of nodal curves and to "translate" a connected subvariety $B \subset \mathrm{Hilb}(X)$ such that every point of B represents a nodal connected curve on X to a connected subvariety $B' \subset \mathrm{Hilb}(X)^{sm}$. For that purpose, we will need a space $\mathbb{T}$ of teeth such that for *every* connected nodal curve C_0 on X, there exist $r > 0$ and $([T_1], \cdots, [T_r]) \in \mathrm{Sym}^r \mathbb{T}$ such that $C := C_0 \cup T_1 \cup \cdots \cup T_r$ is an embedded comb for which the assumptions of Lemma 2.4 hold. When X is src and $\dim X$ is sufficiently large, an appropriate choice of a space of very free rational curves on X will do. When X is src in codimension 1, a space of almost very free rational curves on X is not sufficient. For instance, when $X \to C$ is a src fibration over a curve of positive genus and C_0 has an irreducible component contained in a fiber of the fibration, it is problematic, and in such a case, we will need some more work to construct $\mathbb{T}$. We will explain the details in the following.

Let X be a smooth projective connected variety that is src in codimension 1, and assume that $\dim X \geq 4$. This dimension assumption is a technical condition to work with curves on X rather than mophisms from curves to X, and it may always be achieved by replacing X by $X \times \mathbb{P}^n$ for n large. (There is no harm of doing this because we will eventually be interested in stable birational properties of X.)

Fix a sufficiently ample linear system on X and let $\mathbb{H} \subset \mathrm{Hilb}(X)$ be a space of complete intersection curves by divisors in the system, with a section s such that for all $[H] \in \mathbb{H}$, H is smooth at $p_H := s([H])$. Let $\mathbb{R} \subset \mathrm{Hilb}(X)$ be an irreducible component of the space of almost very free rational curves on X. For a non-negative integer n, define $\mathbb{T}$ to be the constructible subset of $\mathrm{Hilb}(X)$ of all embedded combs T with handles $[H] \in \mathbb{H}$ and teeth $([R_1], \cdots, [R_n]) \in \mathrm{Sym}^n \mathbb{R}$, which are smooth at $p_T := p_H$, such that [14, (42.2)-(42.5)] hold: we have

(1) for any $x \in X$ and $v \in T_X \otimes \kappa(x)$, there exists $[T] \in \mathbb{T}$ such that $x = p_T$ and v is a tangent vector to T at p_T;
(2) for all $[T] \in \mathbb{T}$, we have $H^1(T, N_T(-p_T)) = 0$;

and other more subtle positivity conditions. The space $\mathbb{T}$ is irreducible, and one can deduce from Lemma 2.4 and the ampleness of the linear system defining $\mathbb{H}$ that for sufficiently large n, $\mathbb{T}$ is non-empty; see [14, Theorem 44]. We will fix such n in what follows. Note that $\mathbb{H}$ alone satisfies (1) but not (2) and that $\mathbb{R}$ (with appropriate choice of a section) alone satisfies (2) but not (1).

Now let $B \subset \mathrm{Hilb}(X)$ be a subvariety such that every point of B represents a nodal connected curve on X. For a non-negative integer r, define $\mathrm{EComb}(B, r\mathbb{T}) \subset \mathrm{Hilb}(X)$ to be the constructible set of all 3-nodal combs C with handles $[C_0] \in B$, teeth $([T_1], \cdots, [T_r]) \in \mathrm{Sym}^r \mathbb{T}$, and intersection points $C_0 \cap T_i = \{p_{T_i}\}$.

We say that such C is *balanced* if letting c_0 be the number of irreducible components of C_0, each irreducible component of C_0 has at least $r/2c_0$ teeth attached. Finally, define

$$\mathrm{EComb}(B, r\mathbb{T})^\circ := \{[C] \in \mathrm{EComb}(B, r\mathbb{T}) \mid H^1(C, N_C) = 0 \text{ and } C \text{ is balanced}\}.$$

Since 3-nodal curves are unobstructed, we have $\mathrm{EComb}(\mathrm{B}, \mathrm{r}\mathbb{T})^\circ \subset \mathrm{Hilb}(X)^{sm}$.

By Lemma 2.4, for sufficiently large $r > 0$, $\mathrm{EComb}(B, r\mathbb{T})^\circ$ is non-empty and the forgetful map

$$\Phi\colon \mathrm{EComb}(B, r\mathbb{T})^\circ \to B$$

is surjective. In fact, it has the following stronger properties.

Theorem 2.5 (Kollár–Tian, [14, Theorem 48], Special Case)**.** *Let X be a smooth projective connected variety that is src in codimension 1, and assume that $\dim X \geq 4$. Let $B \subset \mathrm{Hilb}(X)$ be a subvariety such that every point of B represents a nodal connected curve on X. For sufficiently large $r > 0$, let*

$$\Phi\colon \mathrm{EComb}(B, r\mathbb{T})^\circ \to B$$

be the forgetful map. The following hold.

- *(1) Φ has non-empty connected fibers.*
- *(2) Φ satisfies the curve lifting property, That is, every morphism from a 2-pointed irreducible (not neccesarily proper) curve to B admits a lift along Φ.*

In particular, if B is connected, then $\mathrm{EComb}(B, r\mathbb{T})^\circ$ is connected and lies in a single irreducible component of $\mathrm{Hilb}(X)^{sm}$.

Proof. (1) Let $[C_0] \in B$. Then the irreducible components of $\Phi^{-1}([C_0])$ are precisely the closures of the sets of combs with handle C_0 and r teeth from $\mathbb{T}$ where the number of teeth attached to each irreducible component of C_0 is constant. Passing a tooth from one irreducible component of C_0 to another then shows that $\Phi^{-1}([C_0])$ is connected. Here it is crucial that for every nodal $[D] \in \mathrm{Ecomb}(B, (r-1)\mathbb{T})$ and every node $p \in D$, there exists $[T] \in \mathbb{T}$ such that the 3-nodal comb $C := D \cup T$ with $p = p_T$ belongs to $\mathrm{Ecomb}(B, r\mathbb{T})$, and that moreover if $[D] \in \mathrm{Ecomb}(B, (r-1)\mathbb{T})^\circ$, then the exact sequence

$$0 \to N_C(-p)|_T \to N_C \to N_C|_D \to 0$$

together with $H^1(D, N_D) = H^1(T, N_T(-p_T)) = 0$ shows $H^1(C, N_C) = 0$, hence we get $[C] \in \mathrm{Ecomb}(B, r\mathbb{T})^\circ$.

(2) Attaching teeth and choosing a point on teeth yield a desired lift. □

Finally, we generalize Theorem 2.5 over arbitrary perfect fields. Let k be perfect field, X/k be a smooth projective geometrically connected variety such that $X_{\overline{k}}$ is src in codimension 1, and assume that $\dim X \geq 4$. We construct the spaces $\mathbb{H}, \mathbb{R}/k$ as before. By construction, $\mathbb{H}$ is geometrically irreducible; $\mathbb{R}$ is irreducible, but it might not be geometrically so. Now we construct $\mathbb{T}/k$ in the same way as before, except that we make sure to attach the same number of teeth from each geometric component of $\mathbb{R}$ so that $\mathbb{T}$ is defined over k and geometrically irreducible. We then construct $\mathrm{EComb}(B, r\mathbb{T})/k$ and $\mathrm{EComb}(B, r\mathbb{T})^\circ/k$.

We strengthen Theorem 2.5 as follows.

Theorem 2.6 (Kollár–Tian, [14, Theorem 48])**.** *Let k be a perfect field and X/k be a smooth projective geometrically connected variety such that $X_{\overline{k}}$ is src in codimension 1, and assume that $\dim X \geq 4$. Let $B \subset \mathrm{Hilb}(X)$ be a subvariety such that every point of B represents a nodal connected curve on X. For sufficiently large $r > 0$, let*

$$\Phi\colon \mathrm{EComb}(B, r\mathbb{T})^\circ \to B$$

be the forgetful map. The following hold.

- *(1) Φ has non-empty geometrically connected fibers.*
- *(2) Φ satisfies the curve lifting property.*
- *(3) Φ is surjective on K-points for every $k \subset K \subset \overline{k}$.*

In particular, if B is geometrically connected, then $\mathrm{EComb}(B, r\mathbb{T})^\circ$ is geometrically connected and lies in a single geometrically irreducible component of $\mathrm{Hilb}(X)^{sm}$.

Proof. Only (3) needs to be proved. For simplicity, we only consider the case $K = k$, and let $[C_0] \in B(k)$. We would hope to choose a closed point $[C_0 \cup T_1 \cup \cdots \cup T_r] \in \Phi^{-1}([C_0])$ such that the comb

$$C_0 \cup \left(\cup_{i \in \{1, \cdots, r\},\, \sigma \in \mathrm{Gal}(\overline{k}/k)} T_i^\sigma \right),$$

which is defined over k, belongs to $\mathrm{Ecomb}(B, r'\mathbb{T})^\circ$ for some $r' \geq r$. To achieve this, we need to avoid a situation where C_0 and two or more distinct orbits $T_i^\sigma, T_i^{\sigma'}, \cdots$ all intersect at p_i for some i, and such a situation occurs when $p_i \in C_0$ and $[T_i] \in \mathbb{T}$ have different residue fields. To address the issue, we apply Lemma 2.7 below to the projection $\Phi^{-1}([C_0]) \to \mathrm{Sym}^r C_0$. In addition, one can bound r' uniformly; see [14, Lemma 63]. Hence the assertion follows. ☐

Lemma 2.7 (Poonen, [17, Corollary 2])**.** *Let k be a perfect field and $f \colon V \to W$ be a morphism of k-varieties such that $\dim \overline{f(V)} \geq 1$. Then $\{v \in V \mid \kappa(v) = \kappa(f(v))\}$ is Zariski dense in V.*

3. Geometric part of Theorem 1.2 and descending algebraic equivalence for 1-cycles

3.1. Descending algebraic equivalence for 1-cycles.

Definition 3.1. Let k be a perfect field, X/k be a smooth projective geometrically connected variety, and $\alpha_1, \alpha_2 \in CH_i(X)$. We say that α_1 and α_2 are *algebraically equivalent* and write $\alpha_1 \sim_a \alpha_2$ if there exists a geometrically irreducible variety B/k, smooth k-points b_1, b_2 of B, and $\Gamma \in CH_{i+\dim B}(B \times X)$ such that

$$\alpha_1 - \alpha_2 = \Gamma_{b_1} - \Gamma_{b_2} \in CH_i(X).$$

We denote by $A_i(X) := CH_i(X)/\sim_a$.

In Definition 3.1, a pair of smooth k-points may be replaced by a 0-cycle of degree 0: if there exists a geometrically irreducible variety B/k, a 0-cycle $\beta = \sum n_i p_i$ of degree 0 on B^{sm}, and $\Gamma \in CH_{i+\dim B}(B \times X)$, then $\Gamma_\beta = \sum n_i \Gamma_{p_i} \in CH_i(X)$ is algebraically equivalent to 0; see [1, Lemma 3.9].

We will aim to descend algebraic equivalence for 1-cycles on X/k such that $X_{\overline{k}}$ is src in codimension 1. To make our situation more reasonable, we will make the following assumption on k:

($\sharp$) Every geometrically irreducible k-variety has a 0-cycle of degree 1.

If k satisfies ($\sharp$), then for any family $S \to B$ of curves on X with B geometrically irreducible and any $b \in B(\overline{k})$, $S_b \in A_1(X_{\overline{k}})$ lies in the image of $A_1(X) \to A_1(X_{\overline{k}})$.

It is important for us that $k = \mathbb{F}_q$ satisfies ($\sharp$): for a geometrically irreducible $\mathbb{F}_q$-variety V, by Lang–Weil, we have $|V(\mathbb{F}_q)| \sim q^{\dim V}$, hence V has rational points over extensions of coprime degrees, which by corestriction give a zero-cycle of degree 1.

Theorem 3.2 (Kollár–Tian, [14, Theorem 7])**.** *Let k be a perfect field for which ($\sharp$) holds. Let X/k be a smooth projective geometrically connected variety such that $X_{\overline{k}}$ is src in codimension 1. Then*

$$A_1(X) \xrightarrow{\sim} A_1(X_{\overline{k}})^{\mathrm{Gal}(\overline{k}/k)}.$$

For Theorem 3.2, ($\sharp$) is necessary. Indeed, let k be a perfect field, C/k be a smooth projective geometrically connected curve, and $X := C \times \mathbb{P}^1$. Choosing $p \in C(\overline{k})$, if there exists a 1-cycle α on X such that $\alpha_{\overline{k}} \sim_a p \times_{\overline{k}} \mathbb{P}^1_{\overline{k}}$, then $\alpha \cdot (C \times \{0\})$ yields a 0-cycle of degree 1 on C.

The proof of Theorem 3.2 relies on a certain general statement about deformation equivalence for curves, a notion closely related to algebraic equivalence for 1-cycles.

Definition 3.3. Let k be a perfect field and X/k be a smooth projective geometrically connected variety. A *deformation equivalence* of curves mapping to X is a diagram

$$\begin{array}{ccccc} S_{b_i} & \subset & S & \xrightarrow{\pi} & B \times X \\ \downarrow & & \downarrow & & \downarrow \\ b_i & \in & B & = & B \end{array}$$

such that

(1) B is a geometrically connected curve with smooth k-points b_i, and
(2) $S \to B$ is a flat proper morphism of pure relative dimension 1.

The k-points $b_i \in B$ and the closed points of B which correspond to the intersection points of the geometric irreducible components are called the *pivot points*. A deformation equivalence is called *nodal* if all fibers of $S \to B$ are nodal, and it is called 3-*nodal* if all but finitely many fibers of $S \to B$ are nodal and the pivot fibers have only nodes and 3-nodes.

For 1-cycles α_1, α_2 on X, $\alpha_1 \sim_a \alpha_2$ if and only if there exists an effective 1-cycle β on X such that $\alpha_1 + \beta$ and $\alpha_2 + \beta$ are effective and deformation equivalent to each other by a geometrically irreducible base curve B; see [10, Example 10.3.3]. When k is not algebraically closed, deformation equivalence does not necessarily imply algebraic equivalence. For instance, when B is given by

$$B := \{(s^2 - t^2)(u^2 + v^2) = 0\} \subset \mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}} = \operatorname{Proj} \mathbb{Q}[s,t] \times \operatorname{Proj} \mathbb{Q}[u,v],$$

there is a priori no way to achieve algebraic equivalence for the fibers over $([1 : \pm 1], [1 : 0])$.

Theorem 3.4 (Kollár–Tian, [14, Theorem 1 and Theorem 2])**.** *Let k be a perfect field, and let $k \subset L \subset \overline{k}$ be an intermediate field extension. Let X/k be a smooth projective geometrically connected variety that has a 0-cycle of degree 1, and let $\pi_i \colon C_i \to X_L$ be finitely many morphisms from nodal L-curves to X_L such that $((\pi_i)_*[C_i])_{\overline{k}}$ are algebraically equivalent to each other. Then there is a nodal deformation equivalence over $\overline{k}$ with connected fibers*

$$\begin{array}{ccccc} (C_i)_{\overline{k}} \uplus R_i & \subset & S & \xrightarrow{\pi} & B \times_{\overline{k}} X_{\overline{k}} \\ \downarrow & & \downarrow & & \downarrow \\ b_i & \in & B & = & B \end{array}$$

such that

(1) $\pi|_{(C_i)_{\overline{k}} \uplus R_i} \colon (C_i)_{\overline{k}} \uplus R_i \to X_{\overline{k}}$ are defined over L;
(2) $\pi|_{(C_i)_{\overline{k}}} \colon (C_i)_{\overline{k}} \to X_{\overline{k}}$ are isomorphic to the base changes of $\pi_i \colon C_i \to X_L$;
(3) $\pi|_{R_i} \colon R_i \to X_{\overline{k}}$ are defined and isomorphic to each other over k.

The original statement of [14, Theorem 2] assumes that $\chi(C_i, \mathcal{O}_{C_i})$ does not depend on i instead of assuming that X has a 0-cycle of degree 1, and it implies Theorem 3.4; see Lemma 3.7. Theorem 3.4 is stated in this way because it is sufficient for our applications.

A subtle point is that B in Theorem 3.4 tends to be reducible. As noted earlier, algebraic equivalence implies, up to adding a constant cycle, deformation equivalence with B irreducible. Hence Theorem 3.4 achieves an arithmetically useful statement in exchange of getting into a geometrically more complicated situation, in some sense. When $X_{\overline{k}}$ is src in codimension 1, one can remedy this situation by applying Theorem 2.6, which leads to the proof of Theorem 3.2.

3.2. **Proof of Theorem 3.4.** The proof involves constructing new deformation equivalences from given ones, and we summarize relevant tricks (i)-(iii) below. Let k be an algebraically closed field and X be a smooth projective connected variety.

(i) **Gluing deformation equivalences**: Before gluing, the following basic operations are useful. Let $S \to B$ be a deformation equivalence.

- Base changes: Let B' be another connected curve with smooth marked points b'_i, and let $B' \to B$ be a morphism which maps b'_i to b_i. Then the pull-back $S' := S \times_B B' \to B'$ is a deformation equivalence.
- Semistable reduction: There exist a base change $B' \to B$, which is étale over the singular points of B, and a new family $S' \to B'$, where the fibers over smooth points of B' are nodal and the fibers over the singular points of B are unchanged.

- Getting a section: If $S \to B$ has connected fibers, choose smooth points $p_i \in S_{b_i}$ and a sufficiently ample divisor $B' \subset S$ going through p_i. Then $S' := S \times_B B' \to B'$ admits a section. If p_i belong to different irreducible components of S, then B' is reducible.
- Working over the same base: If $S^j \to B^j$ $(j = 1, 2)$ are deformation equivalences, then one can base-change to a sufficiently ample divosor $B \subset B_1 \times B_2$ passing through (b_i^1, b_i^2) to get $S^j \times_{B_j} B \to B$.

Now let

$$B \leftarrow S^j \xrightarrow{\pi^j} B \times X$$

be two deformation equivalences with sections $\sigma^j : B \to S^j$. If $\pi^1 \circ \sigma^1 = \pi^2 \circ \sigma^2$, then we can glue these deformation equivalences along $S^1 \supset \sigma^1(B) \cong B \cong \sigma^2(B) \subset S^2$ to get a new deformation equivalence

$$B \leftarrow S^1 \cup_\sigma S^2 \xrightarrow{\pi} B \times X$$

whose fiber over $b \in B$ is $S_b^1 \cup_{\sigma(b)} S_b^2$.

Otherwise, we introduce auxiliary deformation equivalences called H^{ci}*-families* as follows. Let $\tau^j := pr_X \circ \pi^j \circ \sigma^j : B \to X$, and $\widetilde{\tau}^j : B \to X \times \mathbb{P}^1$ be lifts of τ^j such that $\widetilde{\tau}^1(b) = \widetilde{\tau}^2(b)$ happens only at finitely many smooth points b of B. We then get $g := (\widetilde{\tau}_1, \widetilde{\tau}_2) : B \to \mathrm{Hilb}_2(X \times \mathbb{P}^1)$, where $\mathrm{Hilb}_2(X \times \mathbb{P}^1)$ is the Hilbert scheme of length 2 subschemes of $X \times \mathbb{P}^1$. Let $|H|$ be a sufficiently ample linear system on X, $|H^*| := p_1^*|H| + p_2^*|\mathcal{O}_{\mathbb{P}^1}(2)|$, where $p_1 : X \times \mathbb{P}^1 \to X$ and $p_2 : X \times \mathbb{P}^1 \to \mathbb{P}^1$ are the projections, and a H^{ci}-*curve* will mean a smooth curve L on X of the form $L = H_1 \cap \cdots \cap H_{d-1} \cap H_d^*$ with $H_1, \cdots, H_{d-1} \in p_1^*|H|$ and $H_d^* \in |H^*|$. For each i, let L_i be a H^{ci}-curve passing through $g(b_i)$. Then there exists a deformation equivalence

$$\begin{array}{ccccc} L_i & \subset & \mathcal{H} & \xrightarrow{\pi_{\mathcal{H}}} & B \times X \\ \downarrow & & \downarrow & & \downarrow \\ b_i & \in & B & = & B. \end{array}$$

with two sections $\sigma_{\mathcal{H}}^j : B \to \mathcal{H}$ such that

- $\pi_{\mathcal{H}} \circ \sigma_{\mathcal{H}}^j = \pi^j \circ \sigma^j$, and
- all but finitely many fibers are H^{ci}-curves.

See [14, Corollary 29]. We can then glue the three deformation equivalences $S^j \to B$ and $\mathcal{H} \to B$ along σ^j and $\sigma_{\mathcal{H}}^j$.

By applying these techniques, one can prove the following result.

Proposition 3.5 (Kollár–Tian, [14, Theorem 32])**.** *Let $\pi_i : C_i^1 \cup C_i^2 \to X$ $(i = 1, 2)$ be morphisms from nodal curves to X. Assume that C_i^1, C_i^2 are connected, $|C_1^1 \cap C_1^2| = |C_2^1 \cap C_2^2|$ (which we call r), and we have 3-nodal deformation equivalences*

$$\begin{array}{ccccc} C_i^j & \subset & S^j & \xrightarrow{\pi^j} & B^j \times X \\ \downarrow & & \downarrow & & \downarrow \\ b_i^j & \in & B^j & = & B^j. \end{array}$$

Then there exists a 3-nodal deformation equivalence

$$\begin{array}{ccccc} C_i^1 \uplus (\sqcup_{u=1}^r L_{iu}) \uplus C_i^2 & \subset & S & \xrightarrow{\pi} & B \times X \\ \downarrow & & \downarrow & & \downarrow \\ b_i & \in & B & = & B, \end{array}$$

glued from $S^j \to B^j$ and H^{ci}-families $\mathcal{H}_u \to B$ $(u = 1, \cdots, r)$.

(ii) **Modifying deformation equivalences**

Proposition 3.6 (Kollár–Tian, [14, Theorem 34])**.** *Let*

$$\begin{array}{ccccc} C_i \uplus D_i & \subset & S & \xrightarrow{\pi} & B \times X \\ \downarrow & & \downarrow & & \downarrow \\ b_i & \in & B & = & B \end{array}$$

be a 3*-nodal deformation equivalence. Assume that* C_i, D_i *are connected,* $\chi(D_1) = \chi(D_2)$*, and* $\pi_*[D_1] = \pi_*[D_2]$*. Then there exists a* 3*-nodal deformation equivalence*

$$\begin{array}{ccccc} C_i \uplus R_i & \subset & S' & \xrightarrow{\pi'} & B' \times X \\ \downarrow & & \downarrow & & \downarrow \\ b_i' & \in & B' & = & B', \end{array}$$

glued from $S \to B$*, deformation equivalences over* $\mathrm{Supp}(\pi(D_1)) = \mathrm{Supp}(\pi(D_2))$*,* $C_i \times B' \to B'$*, and* H^{ci}*-families, such that* $\pi'|_{R_i} \colon R_i \to X$ *are isomorphic to each other,* R_i *are connected, and* $C_i \cap R_i = C_i \cap D_i$*.*

Let $E := \mathrm{Supp}(\pi(D_1)) = \mathrm{Supp}(\pi(D_2))$. A crucial step in the proof of Proposition 3.6 is to deform $D_1 \to E$ to $D_2 \to E$ in the space of morphisms from curves to E, and this forces B' to be reducible and requires the χ-independence condition for D_1, D_2.

To address the χ-independence condition, the following lemma will be useful.

Lemma 3.7. *Let* k *be a perfect field and* X/k *be a smooth projective geometrically connected variety that has a* 0*-cycle of degree* 1*. Let* α_1, α_2 *be* 1*-cycles on* X*. Then there exist a* 1*-cycle* β *on* X *and morphisms* $\pi_i \colon C_i \to X$ *from geometrically connected nodal* k*-curves* C_i *to* X *such that* $(\pi_i)_* C_i = \alpha_i + \beta$ *and* $\chi(C_1) = \chi(C_2)$*.*

Proof. Let β' be a 1-cycle on X such that $\alpha_i + \beta'$ are effective. For sufficiently large n, there are smooth k-curves $C_i' \subset X \times \mathbb{P}^n$ such that letting $\pi_i' \colon C_i' \to X$ be the projections, we get $\pi_*' C_i' = \alpha_i + \beta'$. Choose a smooth point from each connected component of C_i', and let $Z_i \subset C_i'$ be the union. By the assumption and moving lemma, there exist closed points $p_1, \cdots, p_a, q_1, \cdots, q_b \in X$ such that $p_1, \cdots, p_a \notin \mathrm{Supp}(\alpha_1 + \beta')$, $q_1, \cdots, q_b \notin \mathrm{Supp}(\alpha_2 + \beta')$, and

$$\chi(C_1') - \deg(Z_1) - \sum \deg(p_u) = \chi(C_2') - \deg(Z_2) - \sum \deg(q_v).$$

Let $D \subset X \times \mathbb{P}^n$ be a smooth complete intersection curve that transversally intersects C_i' at Z_i, $p_u \times \mathbb{P}^1$ at $p_u \times \{0, \infty\}$, and $q_v \times \mathbb{P}^1$ at $q_v \times \{0, \infty\}$. Let

$$C_1 := C_1' \uplus (\sqcup_u p_u \times \mathbb{P}^1) \uplus D, \qquad C_2 := C_2' \uplus (\sqcup_v q_v \times \mathbb{P}^1) \uplus D,$$

and let $\pi_i \colon C_i \to X$ be the projections. Then C_1, C_2 satisfy the desired properties, with $\beta := \beta' + (\pi_i)_* D$. □

(iii) **Lifting a 3-nodal deformation to a nodal deformation**: Every flat deformation of a 3-nodal curve C in X may be lifted to a flat deformation of a morphism from a nodal curve $\widetilde{C}$ to C. More precisely, the map

$$\mathrm{Mor}(\text{Nodal curves}, X) \dashrightarrow \mathrm{Hilb}(X)$$

restricts to an isomorphism over the locus of 3-nodal curves in $\mathrm{Hilb}(X)$. See [14, Example 19] for the details.

We now prove Theorem 3.4.

Proof of Theorem 3.4. Let $k \subset L \subset \overline{k}$, X/k, $\pi_i \colon C_i \to X_L$ as in the statement. For simplicity, we only prove the case when we have two curves π_1 and π_2. By Lemma 3.7, we may assume that C_i are geometrically connected and $\chi(C_i)$ is independent of i.

First assume $k = \overline{k}$. Since $(\pi_1)_*[C_1] \sim_a (\pi_2)_*[C_2]$, the semistable reduction shows that there exists a nodal deformation equivalence $B \leftarrow T \to X$ such that $T_{b_1} - T_{b_2} = C_1 - C_2$. By Proposition 3.5, one can glue 3-nodal equivalences $B \leftarrow T \to B \times X$, $B \leftarrow B \times C_i \to B \times X$, and H^{ci}-curves to get a new 3-nodal deformation equivalence

$$\begin{array}{ccc} C_1 \uplus L_1^i \uplus T_{b_i} \uplus L_2^i \uplus C_2 & \subset & S \xrightarrow{\pi} B \times X \\ \downarrow & & \downarrow \qquad \downarrow \\ b_i & \in & B \;=\!=\!=\; B. \end{array}$$

Let

$$D_1 := L_1^1 \uplus T_{b_1} \uplus L_2^1 \uplus C_2, \qquad D_2 := C_1 \uplus L_1^2 \uplus T_{b_2} \uplus L_2^2.$$

Then $S_{b_1} = C_1 \uplus D_1$, $S_{b_2} = C_2 \uplus D_2$ are 3-nodal deformation equivalent, D_1, D_2 are connected, $\chi(D_1) = \chi(D_2)$, and $\pi_*[D_1] = \pi_*[D_2]$. One can then apply Proposition 3.6 to get a 3-nodal family as desired, and the trick (iii) eliminates 3-nodes.

In the general case, one replaces $(C_i)_{\overline{k}} \uplus R_i$ by their Galois conjugates so that the conjugates of R_i intersect $(C_i)_{\overline{k}}$ at different points, and we achieve this by Lemma 2.7. □

3.3. **Proof of Theorem 3.2.**

We combine Theorem 2.6 and Theorem 3.4 to prove Theorem 3.2.

Proof of Theorem 3.2. As for the injectivity, let $\alpha_1, \alpha_2 \in A_1(X)$ and assume that $\alpha_{1,\overline{k}} \sim_a \alpha_{2,\overline{k}}$. We want to show $\alpha_1 \sim_a \alpha_2$. By adding a constant cycle to α_1 and α_2, we may assume that $\alpha_1 = C_1$ and $\alpha_2 = C_2$ for some curves C_1, C_2 in X. For sufficiently large n, let $\widetilde{C}_1, \widetilde{C}_2$ be nodal curves in $X \times \mathbb{P}^n$ which lift C_1, C_2.

By Theorem 3.4 for $L = k$, there exist a nodal curve R/k and a nodal deformation equivalence $B \leftarrow S \to B \times (X \times \mathbb{P}^n)$ such that

$$[\widetilde{C}_1 \uplus R], [\widetilde{C}_2 \uplus R] \in B(k).$$

By Theorem 2.6, there exist $r > 0$ and $([T_1], \cdots, [T_r]), ([T_1'], \cdots, [T_r']) \in (\mathrm{Sym}^r\, \mathbb{T})^{sm}(k)$ such that

$$[\widetilde{C}_1 \uplus R \uplus (\cup_{i=1}^r T_i)], [\widetilde{C}_2 \uplus R \uplus (\cup_{i=1}^r T_i')] \in \mathrm{EComb}(B, r\mathbb{T})^\circ(k),$$

and $\mathrm{EComb}(B, r\mathbb{T})^\circ$ lies in a geometrically irreducible component of $\mathrm{Hilb}(X \times \mathbb{P}^n)^{sm}$. We get

$$\widetilde{C}_1 + R + \sum T_i \sim_a \widetilde{C}_2 + R + \sum_i T_i'.$$

Since $(\mathrm{Sym}^r\, \mathbb{T})^{sm}$ is geometrically irreducible, we get $\widetilde{C}_1 \sim_a \widetilde{C}_2$, hence $C_1 \sim_a C_2$ as desired. This establishes the injectivity.

As for the surjectivity, let $\beta \in A_1(X_{\overline{k}})^{\mathrm{Gal}(\overline{k}/k)}$ and we aim to show that β lies in the image of $A_1(X) \to A_1(X_{\overline{k}})^{\mathrm{Gal}(\overline{k}/k)}$. By adding to β a constant cycle defined over k, we may assume that $\beta = C$ for some curve C in $X_{\overline{k}}$. For sufficiently large n, let $\widetilde{C}$ be a nodal curve in $X_{\overline{k}} \times_{\overline{k}} \mathbb{P}^n_{\overline{k}}$ which lifts C. Such $\widetilde{C}$ automatically satisfies $\widetilde{C} \in A_1(X_{\overline{k}} \times_{\overline{k}} \mathbb{P}^n_{\overline{k}})^{\mathrm{Gal}(\overline{k}/k)}$, i.e., the Galois conjugates $\widetilde{C}^\sigma \subset X_{\overline{k}} \times_{\overline{k}} \mathbb{P}^n_{\overline{k}}$ are algebraically equivalent to each other. Indeed, by the projective bundle formula, the surjection $A_1(X_{\overline{k}} \times_{\overline{k}} \mathbb{P}^n_{\overline{k}}) \twoheadrightarrow A_1(X_{\overline{k}})$ of $\mathrm{Gal}(\overline{k}/k)$-modules splits, hence $A_1(X_{\overline{k}} \times_{\overline{k}} \mathbb{P}^n_{\overline{k}})^{\mathrm{Gal}(\overline{k}/k)} \twoheadrightarrow A_1(X_{\overline{k}})^{\mathrm{Gal}(\overline{k}/k)}$, and moreover $\ker(A_1(X_{\overline{k}} \times_{\overline{k}} \mathbb{P}^n_{\overline{k}}) \to A_1(X_{\overline{k}})) = A_0(X_{\overline{k}}) = \mathbb{Z}$ is $\mathrm{Gal}(\overline{k}/k)$-invariant.

By Theorem 3.4 for $L = \overline{k}$, there exists a nodal curve R'/k such that $\widetilde{C}^\sigma + R'_{\overline{k}}$ are nodal deformation equivalent to each other, hence they belong to the same geometric connected component B'/k of the locus of $\mathrm{Hilb}(X)$ parametrizing nodal curves. Here we have used that for a variety H/k a connected component of $H_{\overline{k}}$ containing the Galois conjugates of some point of $H_{\overline{k}}$ is defined over k.

By Theorem 2.6, there exist $r > 0$ and $([T_1], \cdots, [T_r]) \in (\mathrm{Sym}^r\, \mathbb{T})^{sm}(k)$ such that

$$[\widetilde{C}^\sigma \uplus R' \uplus (\cup_{i=1}^r T_i)] \in \mathrm{EComb}(B, r\mathbb{T})^\circ(\overline{k}) \text{ for all } \sigma,$$

and $\mathrm{EComb}(B, r\mathbb{T})^\circ$ lies in a geometrically irreducible component Z/k of $\mathrm{Hilb}(X)^{sm}$. By ($\sharp$), there exist zero-cycles γ, δ of degree 1 on Z and $(\mathrm{Sym}^r\, \mathbb{T})^{sm}$, and letting $C_\gamma, C_\delta \in A_1(X \times \mathbb{P}^n)$ be the corresponding 1-cycles,

$$C_\gamma - C_\delta - R' \in A_1(X \times \mathbb{P}^n)$$

maps to $\widetilde{C} \in A_1(X_{\overline{k}} \times_{\overline{k}} \mathbb{P}^n_{\overline{k}})^{\mathrm{Gal}(\overline{k}/k)}$. Pushing forward the class to X concludes the proof. □

3.4. **Proof of the geometric part of Theorem 1.3.**

Proof of the geometric part of Theorem 1.3. Let $X/\mathbb{F}_q$ with $X \to C$ be as in Theorem 1.2. We aim to show that the cycle map

$$\overline{\mathrm{cl}}^2 \colon CH^2(X) \otimes \mathbb{Z}_\ell \to H^4(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell(2))^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)}$$

is surjective. By Bloch–Srinivas [4, Theorem 1 (iii)], the notions of homology equivalence and algebraic equivalence agree for 1-cycles on $X_{\overline{\mathbb{F}}_q}$. Hence the cycle map

$$A_1(X_{\overline{\mathbb{F}}_q}) \otimes \mathbb{Z}_\ell \to H^4(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell(2)) \tag{3.1}$$

is injective.

We claim that (3.1) is in fact an isomorphism. Let $C^\circ \subset C$ be the biggest open subset such that $X^\circ := X \times_C C^\circ \to C^\circ$ is smooth. Then $X_{\overline{\mathbb{F}}_q} \setminus X^\circ_{\overline{\mathbb{F}}_q} = \coprod_i F_i$, where F_i are singular fibers of $X_{\overline{\mathbb{F}}_q} \to C_{\overline{\mathbb{F}}_q}$, and by resolution of singularities, we may assume that F_i are simple normal crossing. We have the following commutative diagram:

$$\begin{array}{ccccccc} \bigoplus_i CH_1(F_i) \otimes \mathbb{Z}_\ell & \longrightarrow & CH_1(X_{\overline{\mathbb{F}}_q}) \otimes \mathbb{Z}_\ell & \longrightarrow & CH_1(X^\circ_{\overline{\mathbb{F}}_q}) \otimes \mathbb{Z}_\ell & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \bigoplus_i H^4_{F_i}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell(2)) & \longrightarrow & H^4(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell(2)) & \longrightarrow & H^4(X^\circ_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell(2)). & & \end{array} \tag{3.2}$$

To prove the claim, it is enough for us to show that the left and right vertical arrows of (3.2) are both surjective. Since $X_\eta/\mathbb{F}_q(C)$ is a geometrically rational surface, by Esnault–Wittenberg [9], the left vertical arrow of (3.2) is surjective. As for the right vertical arrow of (3.2), using that fibers of $f \colon X^\circ \to C^\circ$ are smooth geometrically rational and C° is affine, the Leray spectral sequence for $f_{\overline{\mathbb{F}}_q} \colon X_{\overline{\mathbb{F}}_q} \to C_{\overline{\mathbb{F}}_q}$ yields

$$H^4(X^\circ_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell(2)) = H^0(C^\circ_{\overline{\mathbb{F}}_q}, R^4(f_{\overline{\mathbb{F}}_q})_*\mathbb{Z}_\ell(2)) = \mathbb{Z}_\ell.$$

Since $f_{\overline{\mathbb{F}}_q}$ has a section by de Jong–Starr [7], the right vertical arrow of (3.2) is surjective. This proves the claim.

Taking the Galois invariant part of the isomorphism (3.1) and applying Theorem 3.2 yield

$$A_1(X) \otimes \mathbb{Z}_\ell \xrightarrow{\sim} A_1(X_{\overline{\mathbb{F}}_q})^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)} \otimes \mathbb{Z}_\ell \xrightarrow{\sim} H^4(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell(2))^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)}.$$

The cycle map $\overline{\mathrm{cl}}^2$ factors through the above, hence it is surjective. □

4. Arithmetic part of Theorem 1.2 and two coniveau filtrations

4.1. **Two coniveau filtrations.** Let k be an algebraically closed field.

Definition 4.1. Let X be a smooth projective connected variety of dimension d, and $A \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_\ell, \mathbb{Q}_\ell\}$. The *coniveau filtration* is defined by

$$N^cH^n(X, A) := \sum_{Z \subset X} \mathrm{Im}(H^n_Z(X, A) \to H^n(X, A)) = \sum_{Z \subset X} \mathrm{Ker}(H^n(X, A) \to H^n(X \setminus Z, A)),$$

where $Z \subset X$ runs over all closed subvarieties of codimesion $\geq c$. Similarly, the *strong coniveau filtration* is defined by

$$\widetilde{N}^cH^n(X, A) := \sum_{f \colon T \to X} \mathrm{Im}(f_* \colon H^{n-r}(T, A) \to H^n(X, A)),$$

where the sum is over all smooth projective connected varieties T of dimension $d - r$ with $r \geq c$ and morphisms $f \colon T \to X$.

Over the complex numbers, Deligne [8] showed $N^cH^n(X,\mathbb{Q}) = \widetilde{N}^cH^n(X,\mathbb{Q})$ using the theory of mixed Hodge structures, and Benoist–Ottem [2] recently showed that $N^cH^n(X,\mathbb{Z}) \neq \widetilde{N}^cH^n(X,\mathbb{Z})$ may happen. Scavia and I [20] extended these results to the ℓ-adic setting over arbitrary algebraically closed fields.

The case of special interest for us is when $n = 2i-1$ and $c = i-1$ for some i. In this case, $\widetilde{N}^{i-1}H^{2i-1}(X,\mathbb{Z}_\ell)$ is the image of all cylinder homomorphisms for families of codimension i-cycles on X. More precisely, we have:

Lemma 4.2 (Voisin, [26, Proposition 1.3]; see also [20, Lemma 2.4])**.** *Let X be a smooth projective connected variety. Then we have*

$$\widetilde{N}^{i-1}H^{2i-1}(X,\mathbb{Z}_\ell) = \sum_{(B,\Gamma)} \mathrm{Im}(\Gamma_*\colon H_1(B,\mathbb{Z}_\ell) \to H^{2i-1}(X,\mathbb{Z}_\ell)),$$

where (B,Γ) runs over all smooth projective connected varieties B and $\Gamma \in CH^i(B\times X)$.

Proof. Let $d = \dim X$. We note

$$\widetilde{N}^{i-1}H^{2i-1}(X,\mathbb{Z}_\ell) = \sum_{f\colon T\to X} \mathrm{Im}(f_*\colon H^1(T,\mathbb{Z}_\ell) \to H^{2i-1}(X,\mathbb{Z}_\ell)),$$

where the sum is over all smooth projective connected varieties of dimension $d-i+1$ and morphisms $f\colon T\to X$. For any such $f\colon T\to X$, the Poincaré line bundle $\mathcal{P} \in CH^1(\mathrm{Pic}^0_{T/k}\times T)$ gives an isomorphism

$$\mathcal{P}_*\colon H_1(\mathrm{Pic}^0_{T/k},\mathbb{Z}_\ell) \xrightarrow{\sim} H^1(T,\mathbb{Z}_\ell),$$

hence $(\mathrm{id}_T\times f)_*\mathcal{P} \in CH^i(\mathrm{Pic}^0_{T/k}\times X)$ gives

$$((\mathrm{id}_T\times f)_*\mathcal{P})_* = \mathcal{P}_*(\mathrm{id}_T\times f)_*\colon H_1(\mathrm{Pic}^0_{T/k},\mathbb{Z}_\ell) \to H^{2i-1}(X,\mathbb{Z}_\ell),$$

whose image agrees with that of f_*. This shows $\subset$ for the lemma. As for $\supset$, it is enough to pass to resolutions of $\mathrm{Supp}(\Gamma)$ in characteristic zero, and in positive characteristic, one can instead take ℓ'-alterations due to Gabber [12, Theorem 2.1]. □

The case $i \in \{2, d-1\}$ of Lemma 4.2 is the most interesting because the quotients

$$N^1H^3(X,\mathbb{Z}_\ell)/\widetilde{N}^1H^3(X,\mathbb{Z}_\ell), \qquad N^{d-2}H^{2d-3}(X,\mathbb{Z}_\ell)/\widetilde{N}^{d-2}H^{2d-3}(X,\mathbb{Z}_\ell)$$

are stable birational invariants of smooth projective varieties; see [20, Lemma 2.6, Lemma 2.10]. Hence it would be tempting to ask if these groups may be non-zero for separably rationally connected varieties. The following theorem completely resolves this question for the latter group.

Theorem 4.3 (Tian, [24, Theorem 1.20])**.** *Let X be a smooth projective connected variety that is src in codimension 1, and let $d = \dim X$. Then we have*

$$N^{d-2}H^{2d-3}(X,\mathbb{Z}_\ell) = \widetilde{N}^{d-2}H^{2d-3}(X,\mathbb{Z}_\ell).$$

Theorem 4.3 modulo torsion over the complex numbers was previously established by Voisin [26]. We postpone the proof of Theorem 4.3 until Section 6 and Section 7.

4.2. **ℓ-adic Walker Abel–Jacobi maps.** Let $X/\mathbb{F}_q$ be a smooth projective geometrically connected variety. One can show that the restriction of $\mathrm{cl}^i_{\mathrm{AJ}}$ to $CH^i(X)_{\mathrm{alg}}$ factors through the ℓ-adic Walker Abel–Jacobi map

$$\mathrm{cl}^i_{\mathrm{WAJ}}\colon CH^i(X)_{\mathrm{alg}}\otimes\mathbb{Z}_\ell \to H^1(\mathbb{F}_q, N^{i-1}H^{2i-1}(X_{\overline{\mathbb{F}}_q},\mathbb{Z}_\ell(i))).$$

See [20, §4.2] for the construction and the basic properties of $\mathrm{cl}_{\mathrm{WAJ}}$. The complex analogue of $\mathrm{cl}_{\mathrm{WAJ}}$ was first constructed by Walker in [27].

Proposition 4.4 (Scavia–Suzuki, [20, Proposition 5.6])**.** *Let $X/\mathbb{F}_q$ be a smooth projective geometrically connected variety. The image of $\mathrm{cl}_{\mathrm{WAJ}}$ agrees with the image of*

$$H^1(\mathbb{F}_q,\widetilde{N}^{i-1}H^{2i-1}(X_{\overline{\mathbb{F}}_q},\mathbb{Z}_\ell(i))) \to H^1(\mathbb{F}_q,N^{i-1}H^{2i-1}(X_{\overline{\mathbb{F}}_q},\mathbb{Z}_\ell(i))).$$

Proof. A version of Lemma 4.2 shows that

$$\widetilde{N}^{i-1}H^{2i-1}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell(i)) = \sum_{(B,\Gamma)} \mathrm{Im}(\Gamma_* \colon H_1(B_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell) \to H^{2i-1}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell(i))),$$

where (B,Γ) runs over all smooth *geometrically connected* $\mathbb{F}_q$-varieties B and $\Gamma \in CH^i(B \times X)$; see [20, Lemma 2.12]. In addition, results of Achter–Casalaina-Martin–Vial [1] show that

$$CH^i(X)_{\mathrm{alg}} = \sum_{(B,\Gamma)} \mathrm{Im}(\Gamma_* \colon CH_0(B)_{\mathrm{hom}} \to CH^i(X)),$$

where (B,Γ) runs over all B, Γ as above; see [20, Lemma 4.3]. We have the induced diagram:

$$\begin{array}{ccccc}
\oplus_{(B,\Gamma)} CH_0(B)_{\mathrm{hom}} \otimes \mathbb{Z}_\ell & & \xrightarrow{(\Gamma_*)} & & CH^i(X)_{\mathrm{alg}} \otimes \mathbb{Z}_\ell \\
\wr\downarrow \mathrm{cl}_{\mathrm{AJ},0} & & & & \downarrow \mathrm{cl}^i_{\mathrm{WAJ}} \\
\oplus_{(B,\Gamma)} H^1(\mathbb{F}_q, H_1(B_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell)) & & \xrightarrow{(\Gamma_*)} & & H^1(\mathbb{F}_q, N^{i-1}H^{2i-1}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell(i))), \\
& \searrow & & \nearrow & \\
& & H^1(\mathbb{F}_q, \widetilde{N}^{i-1}H^{2i-1}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell(i))) & &
\end{array}$$

where the left vertical isomorphism follows from [6, Théorème 5]. The proof is now immediate. □

The map $\mathrm{cl}^i_{\mathrm{WAJ}}$ may be non-surjective; see [19, Theorems 1.4, 1.5].

4.3. **Proof of the arithmetic part of Theorem 1.3.**

Proof of the arithmetic part of Theorem 1.3. Let $X/\mathbb{F}_q$ with $X \to C$ be as in the statement of Theorem 1.3. We aim to show that the ℓ-adic Abel–Jacobi map

$$\mathrm{cl}^2_{\mathrm{AJ}} \colon CH^2(X)_{\mathrm{hom}} \otimes \mathbb{Z}_\ell \to H^1(\mathbb{F}_q, H^3(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell(2)))$$

is surjective. By Theorem 4.3 for $d = 3$ and Proposition 4.4 for $i = 2$, $\mathrm{cl}^2_{\mathrm{WAJ}}$ is surjective. Moreover, by Merkurjev–Suslin [15], we have $H^3(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell) = N^1H^3(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell)$. Hence $\mathrm{cl}^2_{\mathrm{AJ}}$ is surjective. □

Proof of Theorem 1.3. Since $\overline{\mathrm{cl}}^2$ and $\mathrm{cl}^2_{\mathrm{AJ}}$ are both surjective, by (1.2), we conclude that the cycle map

$$\mathrm{cl}^2 \colon CH^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))$$

is surjective. □

The proof of Theorem 1.3 shows the following more general statement.

Theorem 4.5 (Tian, [24, Theorem 1.11])**.** *Let $X/\mathbb{F}_q$ be a smooth projective geometrically connected variety such that $X_{\overline{\mathbb{F}}_q}$ is src in codimension* 1*, and let $d = \dim X$. Assume that*

(1) the cycle map $A_1(X_{\overline{\mathbb{F}}_q}) \otimes \mathbb{Z}_\ell \to H^{2d-2}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell(d-1))$ is an isomorphism, and
(2) $H^{2d-3}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell) = N^{d-2}H^{2d-3}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell)$.

Then the cycle map $\mathrm{cl}^{d-1} \colon CH_1(X) \otimes \mathbb{Z}_\ell \to H^{2d-2}(X, \mathbb{Z}_\ell(d-1))$ is surjective.

Theorem 4.5(1) and (2) are stable birational properties of $X_{\overline{\mathbb{F}}_q}$; see [24, Lemma 2.36]. It follows that, for every smooth projective geometrically rational $\mathbb{F}_q$-variety X of dimension d, $\mathrm{cl}^{d-1} \colon CH_1(X) \otimes \mathbb{Z}_\ell \to H^{2d-2}(X, \mathbb{Z}_\ell(d-1))$ is surjective.

5. Equi-dimensional families of cycles

Let k be an algebraically closed field.

Definition 5.1. An *equi-dimensional family of i-cycles on a variety X parametrized by a variety B* is a finite $\mathbb{Z}$-linear combination $\Gamma = \sum_j m_j \Gamma_j$, where $m_j \in \mathbb{Z}$ and $\Gamma_j \subset B \times X$ are integral, such that the following conditions hold.

(1) Each Γ_j dominates an irreducible component of B.
(2) Each non-empty fiber of $\Gamma_j \to B$ is pure of dimension i.
(3) The fat point condition [23, Definition 3.1.3] is satisfied (i.e., for each $\Gamma_j \to B$, the limiting fiber along a curve on B is unique).
(4) The field-of-definition condition [23, Lemma 3.3.9 and the following paragraph] is satisfied (i.e., for each $\Gamma_j \to B$, the pull-back along a map from the spectrum of a field to B is well-defined).

For $b \in B$, we denote $\Gamma_b := \sum m_j (\Gamma_j)_b$, where $(\Gamma_j)_b$ is the pull-back of $\Gamma_j \to B$ along $b \hookrightarrow B$. We say that (B, Γ) is *constant* (resp. *constant modulo ℓ^n*) if Γ_b (resp. Γ_b modulo ℓ^n) does not depend on b.

For an equi-dimensional family (B, Γ) of i-cycles and a morphism $f \colon B' \to B$, one can define a pull-back equi-dimensional family $(B', f^*\Gamma)$ of i-cycles; see [23, the paragraph before Lemma 3.3.10]. For equi-dimensional families $(B, \Gamma), (B', \Gamma')$ of i-cycles, a morphism $f \colon (B', \Gamma') \to (B, \Gamma)$ will mean a morphism $f \colon B' \to B$ such that $\Gamma' = f^*\Gamma$.

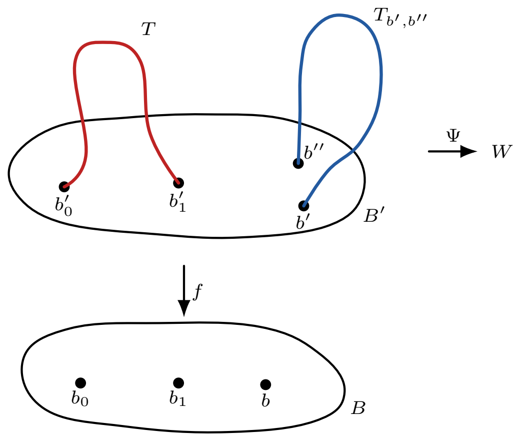
Theorem 5.2 (Tian, [24, Theorem 3.11])**.** *Let X be a smooth projective connected variety that is src in codimension 1. Let (B, Γ_B) be an equi-dimensional family of 1-cycles on X, where B is irreducible; $b_0, b_1 \in B$ such that $(\Gamma_B)_{b_0} = (\Gamma_B)_{b_1}$; $b \in B$ be general. Then there exist:*

(1) an equi-dimensional family $(B', \Gamma_{B'})$ of 1-cycles on X, where B' is normal, quasi-projective, and irreducible; $b'_0, b'_1 \in B'$; a generically finite surjective projective morphism $f \colon (B', \Gamma_{B'}) \to (B, \Gamma_B)$ such that $f(b'_i) = b_i$;
(2) a constant equi-dimensional family (T, Γ_T) of 1-cycles on X and constant equi-dimensional families $(T_{b',b''}, \Gamma_{T_{b',b''}})$ of 1-cycles on X for all $b', b'' \in f^{-1}(b)$, where $T, T_{b',b''}$ are projective and connected; smooth points $t_0, t_1 \in T$ and smooth points $t^{b'}_{b',b''}, t^{b''}_{b',b''} \in T_{b',b''}$;
(3) an equi-dimensional-family (W, Γ_W) of 1-cycles on X, where W is normal (resp. smooth in characteristic zero), projective, and irreducible,

and moreover there exists a morphism

$$\Psi \colon (B', \Gamma_{B'}) \sqcup (T, \Gamma_T) \sqcup (\sqcup_{b',b'' \in f^{-1}(b)} (T_{b',b''}, \Gamma_{b',b''})) \to (W, \Gamma_W)$$

such that $\Psi(b'_i) = \Psi(t_i)$; $\Psi(b') = \Psi(t^{b'}_{b',b''})$, $\Psi(b'') = \Psi(t^{b''}_{b',b''})$; $\Psi(T), \Psi(T_{b',b''}) \subset W^{sm}$.



The proof of Theorem 5.2 is based on Theorem 2.6 and Theorem 3.4. See [24] for the details.

6. Theorem 4.3 in the complex case and Lawson homology

6.1. **Lawson homology.** Let X be a complex projective variety, and fix a very ample line bundle on X. Then the Chow variety $\mathrm{Chow}_i(X)$ of effective i-cycles on X admits a decomposition

$$\mathrm{Chow}_i(X) = \coprod_{j\geq 0} \mathrm{Chow}_{i,j}(X),$$

where $\mathrm{Chow}_{i,j}(X)$ is the Chow variety of effective i-cycles of degree j on X. The varieties $\mathrm{Chow}_{i,j}(X)$ are quite singular, even when X is smooth.

Definition 6.1. Let X be a complex projective variety. We define $\mathcal{Z}_i(X)$ to be the group completion of the monoid $\mathrm{Chow}(X)^{an}$, which is isomorphic to the quotient $\mathrm{Chow}(X)^{an} \times \mathrm{Chow}(X)^{an}/\sim$, where $([a],[b]) \sim ([a+c],[b+c])$ for all $[a],[b],[c] \in \mathrm{Chow}(X)^{an}$. We define the *Lawson homology* of X by

$$L_iH_n(X) := \pi_{n-2i}(\mathcal{Z}_i(X)).$$

Let X be a complex projective variety. For an equidimensional family (B,Γ) of i-cycles on X, there is a well-defined map of topological groups

$$\Gamma_*\colon \mathcal{Z}_0(B) \to \mathcal{Z}_i(X).$$

This induces

$$\Gamma_*\colon L_0H_1(B) = \pi_1(\mathcal{Z}_0(B)) \to L_iH_{2i+1}(X) = \pi_1(\mathcal{Z}_i(X)).$$

If X is smooth, then for any smooth projective variety B and $\Gamma \in CH_{\dim B+1}(B\times X)$, the map $\Gamma_*\colon L_0H_1(B) \to L_1H_3(X)$ may be defined using functoriality of Lawson homology.

Theorem 6.2 (Tian, [24, Theorem 4.12])**.** *Let X be a complex smooth projective connected variety that is src in codimension 1. The map*

$$\bigoplus_{(B,\Gamma)} L_0H_1(B) \xrightarrow{(\Gamma_*)} L_1H_3(X),$$

where (B,Γ) runs over all complex smooth projective varieties B and $\Gamma \in CH_{\dim B+1}(B\times X)$, is surjective.

Let X be a complex projective variety. There is a continuous map $s\colon \mathcal{Z}_i(X)\wedge \mathbb{P}^1 \to \mathcal{Z}_{i-1}(X)$, called the *s-map*, which induces

$$s\colon L_iH_n(X) \to L_{i-1}H_n(X),$$

hence the cycle map

$$L_iH_n(X) \xrightarrow{s^i} L_0H_n(X) \xrightarrow{\sim} H_n(X,\mathbb{Z}),$$

where the last isomorphism is due to Dold–Thom.

The case of special interest for us is when $n = 2i+1$. In this case, $L_iH_{2i+1}(X) = \pi_1(\mathcal{Z}_i(X))$ and the cycle map $L_iH_{2i+1}(X) \to H_{2i+1}(X,\mathbb{Z})$ may be regarded as a cylinder homomorphism. We have the following analogue of Lemma 4.2 for the niveau filtration.

Lemma 6.3 (Walker, [27, Proposition 2.8])**.** *Let X be a complex projective variety. We have*

$$N_{i+1}H_{2i+1}(X,\mathbb{Z}) = \mathrm{Im}(L_iH_{2i+1}(X) \to H_{2i+1}(X,\mathbb{Z})).$$

We can now deduce the complex case of Theorem 4.3.

Corollary 6.4. *Let X be a complex smooth projective connected variety that is src in codimension 1. We have*

$$N_2H_3(X,\mathbb{Z}) = \widetilde{N}_2H_3(X,\mathbb{Z}).$$

Proof. The following diagram commutes

$$\begin{array}{ccc} \bigoplus_{(B,\Gamma)} L_0H_1(B) & \xrightarrow{(\Gamma_*)} & L_iH_{2i+1}(X) \\ \downarrow \wr & & \downarrow \\ \bigoplus_{(B,\Gamma)} H_1(B,\mathbb{Z}) & \xrightarrow{(\Gamma_*)} & H_{2i+1}(X,\mathbb{Z}), \end{array}$$

where (B, Γ) runs over all complex smooth projective varieties B and $\Gamma \in CH_{\dim B+1}(B \times X)$. Then, by Theorem 6.2 and Lemma 6.3, we have

$$N_2H_3(X, \mathbb{Z}) = \sum_{(B,\Gamma)} \mathrm{Im}(\Gamma_*\colon H_1(B, \mathbb{Z}) \to H_3(X, \mathbb{Z})),$$

where (B, Γ) runs over all B, Γ as above, and by Lemma 4.2, the right hand side agrees with $\widetilde{N}_2H_3(X, \mathbb{Z})$. $\square$

6.2. **Proof of Theorem 6.2.** Let X be a complex projective variety. For an equi-dimensional family (B, Γ_B) of i-cycles on X, we denote by $K(B), I(B)$ the kernel and image of $(\Gamma_B)_*\colon \mathcal{Z}_0(B) \to \mathcal{Z}_i(X)$. Then $K(B) \to \mathcal{Z}_0(B) \to I(B)$ is a topological fibration, which induces a long exact sequence of homotopy groups

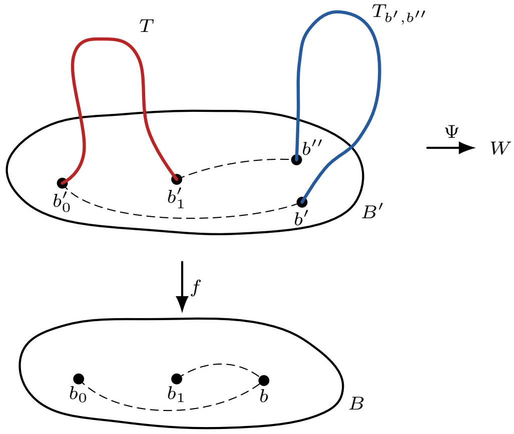
$$\cdots \to \pi_1(K(B)) \to \pi_1(\mathcal{Z}_0(B)) \to \pi_1(I(B)) \to \pi_0(K(B)) \to \cdots .$$

Proof of Theorem 6.2. Let $[L] \in L_1H_3(X) = \pi_1(\mathcal{Z}_1(X))$. We aim to find an equi-dimensional family (W, Γ_W) of 1-cycles on X such that W is smooth projective and $[L]$ lies in the image of

$$(\Gamma_W)_*\colon L_0H_1(W) = \pi_1(\mathcal{Z}_0(W)) \to L_1H_3(X) = \pi_1(\mathcal{Z}_1(X)).$$

To start off, there exist an equi-dimensional family (B, Γ_B) of 1-cycles on X with B normal projective irreducible and a path $[0, 1] \to B$ such that the composition $[0, 1] \to B \to \mathcal{Z}_0(B) \xrightarrow{(\Gamma_B)_*} \mathcal{Z}_1(X)$ represents $[L]$; see [24, Lemma 4.13]. Let $b_0, b_1 \in B$ be the image of $0, 1 \in [0, 1]$. Then $\Gamma_{b_0} = \Gamma_{b_1}$.

We now apply Theorem 5.2 and get $(B', \Gamma_{B'})$, $b'_0, b'_1 \in B'$, $f\colon (B', \Gamma_{B'}) \to (B, \Gamma_B)$ such that $f(b'_i) = b_i$, $(T, \Gamma_T), (T_{b',b''}, \Gamma_{b',b''})$ $(b', b'' \in f^{-1}(b))$, and $\Psi\colon (B', \Gamma_{B'}) \sqcup (T, \Gamma_T) \sqcup (\sqcup_{b',b'' \in f^{-1}(b)}(T_{b',b''}, \Gamma_{b',b''})) \to (W, \Gamma_W)$ as in the theorem. We first assume that $f\colon B' \to B$ is étale over b_0, b_1. We can then lift the path $[0, 1] \to B$ along $(B', \Gamma_{B'}) \to (B, \Gamma_B)$ and close it up using the constant families $(T, \Gamma_T), (T_{b',b''}, \Gamma_{b',b''})$.



We then get a desired pair (W, Γ_W) and a loop on W which maps to $[L]$ under $(\Gamma_W)_*$.

In general, the above assumption is too optimistic to be true. Instead, we use that $f_*\colon \mathcal{Z}_0(B') \to \mathcal{Z}_0(B)$ is a topological fibration, and lift the path $[0, 1] \to B$ along the fibration. For this method to work out, one needs that any two points in $\mathrm{Ker}(f_*)$ are connected by paths in $\mathrm{Ker}(f_*)$ and $T_{b',b''}$, which are constant in $\mathcal{Z}_1(X)$.

Here is a more precise proof. The topological fibrations

$$K(B) \to \mathcal{Z}_0(B) \to I(B), \;\; K(B') \to \mathcal{Z}_0(B') \to I(B'), \;\; K(W) \to \mathcal{Z}_0(W) \to I(W)$$

together with the fact that $(\Gamma_{B'})_*\colon \mathcal{Z}_0(B') \to \mathcal{Z}_1(X)$ factors as

$$\mathcal{Z}_0(B') \xrightarrow{f_*} \mathcal{Z}_0(B) \xrightarrow{(\Gamma_B)_*} \mathcal{Z}_1(X), \qquad \mathcal{Z}_0(B') \xrightarrow{\Psi_*} \mathcal{Z}_0(W) \xrightarrow{(\Gamma_W)_*} \mathcal{Z}_1(X)$$

induce the following commutative diagram of homotopy groups

$$\begin{array}{ccccc} \pi_1(\mathcal{Z}_0(W)) & \longrightarrow & \pi_1(I(W)) & \longrightarrow & \pi_0(K(W)) \\ \uparrow \Psi_* & & \uparrow \Psi_* & & \uparrow \Psi_* \\ \pi_1(\mathcal{Z}_0(B')) & \longrightarrow & \pi_1(I(B')) & \longrightarrow & \pi_0(K(B')) \\ \downarrow f_* & & \wr \downarrow f_* & & \downarrow f_* \\ \pi_1(\mathcal{Z}_0(B)) & \longrightarrow & \pi_1(I(B)) & \longrightarrow & \pi_0(K(B)), \end{array}$$

where $f_* \colon \pi_1(I(B')) \xrightarrow{\sim} \pi_1(I(B))$ because $f_* \colon \mathcal{Z}_0(B') \to \mathcal{Z}_0(B)$ is surjective and thus $I(B') \xrightarrow{\sim} I(B)$. Let $[L'] \in \pi_1(I(B'))$ be a unique lift of $[L] \in \pi_1(I(B))$. It remains to show that $\Psi_*[L'] \in \pi_1(I(W))$ lifts to $\pi_1(\mathcal{Z}_0(W))$, which will imply that for any lift $[L_W] \in \pi_1(\mathcal{Z}_0(W))$ of $\Psi_*[L']$, we have $(\Gamma_W)_*[L_W] = [L]$. To prove the assertion, it is enough for us to show that the image of $[L']$ in $\pi_0(K(W))$ vanishes. Using the topological fibration $\mathrm{Ker}(f_*) \to K(B') \xrightarrow{f_*} K(B)$, we see that the image of $[L']$ in $\pi_0(K(B'))$ is represented by $[b'_0 - b'_1]$ modulo $\pi_0(\mathrm{Ker}(f_*))$. Moreover, the flattening theorem together with the normalness of B, B' shows that $\pi_0(\mathrm{Ker}(f_*))$ is generated by $[b' - b'']$ ($b', b'' \in f^{-1}(b)$); see [24, Claim 4.14]. Finally, since $(T, \Gamma_T), (T_{b',b''}, \Gamma_{b',b''})$ are constant, they map to connected curves in $K(W)$, hence in particular, $[\Psi(b'_0)] = [\Psi(b'_1)]$ and $[\Psi(b')] = [\Psi(b'')]$ ($b', b'' \in f^{-1}(b)$) in $\pi_0(K(W))$. This proves the assertion, concluding the proof of the theorem. □

7. Theorem 4.3 in the general case and higher Chow groups

Let k be an algebraically closed field.

7.1. **Higher Chow groups.** With the full machinery of Lawson homology, Theorem 4.3 in the general case would follow from the same argument as in the complex case. Unfortunately, the s-map is (so far) not available in positive characteristic. As an alternative to Lawson homology, we will use higher Chow groups.

Definition 7.1. Let X be a quasi-projective equi-dimensional variety, and let $d := \dim X$. Let $z^i(X, j)$ be the free abelian group generated by all codimension i subvarieties of $X \times \Delta^j$ which meet all faces $X \times \Delta^{j'}$ properly for all $j' \leq j$, where

$$\Delta^{j'} := \mathrm{Spec}\left(k[t_0, \cdots, t_{j'}]/\left(\sum_{j''=0}^{j'} t_{j''} = 1\right)\right)$$

is the algebraic j'-simplex. Let $z^i(X, *)$ be the associated chain complex, whose boundary maps are alternating sums of restrictions to faces. We define the *higher Chow group* $CH^i(X, j, A)$ to be the j-th homology group of the complex $z^i(X, j) \otimes_{\mathbb{Z}} A$ and define $CH_i(X, j, A) := CH^{d-i}(X, j, A)$.

As in the case of Chow groups, proper push-forwards, flat pull-backs, exterior products, and pull-backs along arbitrary morphisms between smooth varieties are available for higher Chow groups; see [3]. In particular, correspondences act on higher Chow groups of smooth varieties.

We introduce a key complex for higher Chow groups. For a smooth projective variety X, the exact sequence $0 \to \mathbb{Z} \xrightarrow{\ell^n} \mathbb{Z} \to \mathbb{Z}/\ell^n \to 0$ induces a long exact sequence

$$\cdots \to CH_i(X, 1) \xrightarrow{\ell^n} CH_i(X, 1) \to CH_i(X, 1, \mathbb{Z}/\ell^n) \to CH_i(X) \xrightarrow{\ell^n} CH_i(X) \to CH_i(X)/\ell^n \to 0.$$

For a smooth projective variety B and $\Gamma \in CH_{\dim B + i}(B \times X)$, we have a commutative diagram:

$$\begin{array}{ccccc} CH_0(B, 1, \mathbb{Z}/\ell^n) & \twoheadrightarrow & CH_0(B)[\ell^n] & \longrightarrow & A_0(B)[\ell^n] \\ \downarrow \Gamma_* & & \downarrow \Gamma_* & & \downarrow \Gamma_* \\ CH_i(X, 1, \mathbb{Z}/\ell^n) & \twoheadrightarrow & CH_i(X)[\ell^n] & \longrightarrow & A_i(X)[\ell^n]. \end{array}$$

Since $A_0(B) = \mathbb{Z}^{\pi_0(B)}$ is torsion-free, we obtain a complex

$$CH_0(B, 1, \mathbb{Z}/\ell^n) \xrightarrow{\Gamma_*} CH_i(X, 1, \mathbb{Z}/\ell^n) \to A_i(X)[\ell^n].$$

Theorem 7.2 (Tian, [24, Theorem 5.12])**.** *Let X be a smooth projective connected variety that is src in codimension* 1*. The complex*

$$\bigoplus_{(B,\Gamma)} CH_0(B, 1, \mathbb{Z}/\ell^n) \xrightarrow{(\Gamma_*)} CH_1(X, 1, \mathbb{Z}/\ell^n) \to A_1(X)[\ell^n],$$

where (B, Γ) runs over all smooth projective varieties B and $\Gamma \in CH_{\dim B+1}(B \times X)$, is exact.

Theorem 7.2 implies Theorem 6.2 (cf. [24, Remark 5.13]).

Second proof of Theorem 6.2. For a complex smooth projective variety X, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & CH_i(X,1)/\ell^n & \longrightarrow & CH_i(X,1,\mathbb{Z}/\ell^n) & \longrightarrow & CH_i(X)[\ell^n] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \wr & & \downarrow & & \\ 0 & \longrightarrow & L_iH_{2i+1}(X)/\ell^n & \longrightarrow & L_iH_{2i+1}(X,\mathbb{Z}/\ell^n) & \longrightarrow & L_iH_{2i}(X)[\ell^n] & \longrightarrow & 0, \end{array}$$

where the middle vertical arrow is an isomorphism by a theorem of Suslin–Voevodsky, and in addition, we have $L_iH_{2i}(X) = A_i(X)$. Hence, if X is src in codimension 1, by Theorem 7.2, for all ℓ and n, the composition

$$\bigoplus_{(B,\Gamma)} L_0H_1(B) \xrightarrow{(\Gamma_*)} L_1H_3(X) \to L_1H_3(X)/\ell^n,$$

where (B, Γ) runs over all complex projective varieties B and $\Gamma \in CH_{\dim B+i}(B \times X)$, is surjective, hence the cokernel of $\bigoplus_{(B,\Gamma)} L_0H_1(B) \xrightarrow{(\Gamma_*)} L_1H_3(X)$ is divisible. On the other hand, the decomposition of the diagonal shows that there exists N such that the cokernel is N-torsion. Hence $\bigoplus_{(B,\Gamma)} L_0H_1(B) \xrightarrow{(\Gamma_*)} L_1H_3(X)$ is surjective. □

7.2. **Proof of Theorem 7.2.** To make an analogy between Lawson homology and higher Chow groups, we introduce Chow presheaves and sheaves.

Definition 7.3. Let X be a quasi-projective equi-dimensional variety. We denote by $z_{equi}(X, i)$ the presheaf on the category of varieties which associates to a given variety B the free abelian group of equi-dimensional families of i-cycles on X parametrized by B.

For a presheaf $\mathcal{F}$ on the category of varieties, we denote by $C_*\mathcal{F}$ the Suslin complex of $\mathcal{F}$: for a given variety S, $C_i\mathcal{F}(S) = \mathcal{F}(S \times \Delta^i)$ and the boundary maps are alternating sums of restrictions to faces.

Recall that the h-topology is a Grothendieck topology on the category of schemes, which is generated by h-covers, and important examples for us of h-covers are surjective proper morphisms. For a quasi-projective equi-dimensional variety X, let $z_{equi}(X, i)_h$ be the h-sheafification of $z_{equi}(X, i)$.

It will be important to work also with h-sheaves rather than working only with presheaves.

Lemma 7.4. *Let X, X' be projective equi-dimensional varieties and $f : X' \to X$ be a surjective morphism. Then $f_* : z_{equi}(X', 0)_h \to z_{equi}(X, 0)_h$ is a surjection of h-sheaves.*

In the setting of Lemma 7.4, $f_* : z_{equi}(X', 0) \to z_{equi}(X, 0)$ is not necessarily a surjection of presheaves.

Proof of Lemma 7.4. It is important for us that $f : X' \to X$ is a surjective proper morphism, hence an h-cover. By [23, Proposition 4.2.14], $z_{equi}(B, 0)_h$ is isomorphic to the free h-sheaf $\mathbb{Z}[X]$ generated by the sheaf of sets represented by X. Let S be a variety. Then any morphism $S \to X$ lifts along the h-cover $f : X' \to X$ to $S \times_X X' \to X'$. This shows that $f_* : z_{equi}(X', 0)_h \to z_{equi}(X, 0)_h$ is surjective as h-sheaves. □

We can express higher Chow groups in terms of Chow presheaves and sheaves.

Lemma 7.5. *Let X be a quasi-projective equi-dimensional variety. Then*

$$CH_i(X, j, \mathbb{Z}/\ell^n) = H_j(C_*(z_{equi}(X, i) \otimes \mathbb{Z}/\ell^n)(\mathrm{Spec}(k))) = \mathbb{H}_h^{-j}(\mathrm{Spec}(k), C_*(z_{equi}(X, i)_h \otimes \mathbb{Z}/\ell^n)).$$

Proof. The complex $C_*z_{equi}(X, i)(\mathrm{Spec}(k))$ is a subcomplex of $z^{\dim X-i}(X, *)$, and by [21, Theorem 3.2] and [13, Theorem 5.6.4], for all $0 \le i \le \dim X$, the inclusion is a quasi-isomorphism up to inverting the characteristic p of k. Accordingly, $CH_i(X, j)[1/p]$ agrees with the j-th homology group of $C_*z_{equi}(X, i)[1/p](\mathrm{Spec}(k))$, and using that $C_*z_{equi}(X, i)$ and $z^{\dim X-i}(X, *)$ are torsion free, the first equality follows.

As for the second equality, since k is algebraically closed, the hypercohomology spectral sequence shows

$$\mathbb{H}_h^{-j}(\mathrm{Spec}(k), C_*(z_{equi}(X, i)_h \otimes \mathbb{Z}/\ell^n)) = H_h^0(\mathrm{Spec}(k), H^{-j}(C_*(z_{equi}(X, i)_h \otimes \mathbb{Z}/\ell^n))).$$

Moreover, since $z_{equi}(X, i) \otimes \mathbb{Z}/\ell^n$ is a presheaf with transfers, the cohomology presheaves $H^{-j}(C_*(z_{equi}(X, i) \otimes \mathbb{Z}/\ell^n))$ are homotopy invariant presheaves with transfers by [25, Proposition 3.6]. (Recall that a presheaf F with transfers is a presheaf on the category of smooth varieties for which finite correspondences act, and it is called homotopy invariant if for every smooth variety X, we have $F(X \times \mathbb{A}^1) = F(X)$.) By the Suslin rigidity theorem [22, Theorem 4.5] applied to $H^{-j}(C_*z_{equi}(X, i) \otimes \mathbb{Z}/\ell^n)$, we then have

$$H_h^0(\mathrm{Spec}(k), H^{-j}(C_*(z_{equi}(X, i)_h \otimes \mathbb{Z}/\ell^n))) = H^0(\mathrm{Spec}(k), H^{-j}(C_*(z_{equi}(X, i) \otimes \mathbb{Z}/\ell^n))),$$

where the last term is equal to $H_j(C_*(z_{equi}(X, i) \otimes \mathbb{Z}/\ell^n)(\mathrm{Spec}(k)))$. Note that [22, Theorem 4.5] is only proved in characteristic zero, but in positive characteristic, the same proof works after replacing the resolution of singularities by de Jong's alterations, because one only needs that for any variety Y there exist a smooth variety Z and a proper surjective morphism $Z \to Y$. The second equality follows. $\square$

Lemma 7.5 may be compared with the definition of Lawson homology, where the presheaf $z_{equi}(X, i)$ or the h-sheaf $z_{equi}(X, i)_h$ plays an analogous role to the topological group $\mathcal{Z}_i(X)$, where both classify equi-dimensional families of i-cycles on X.

For an equi-dimensional family (B, Γ_B) of i-cycles on X, there is a well-defined map of h-sheaves

$$(\Gamma_B)_* \colon z_{equi}(B, 0)_h \to z_{equi}(X, 1)_h,$$

and let $K(B), I(B)$ be the kernel and image of the map. Since $z_{equi}(B, 0)_h, z_{equi}(X, 1)_h$ are torsion-free, so are $K(B), I(B)$. Accordingly, we get a short exact sequence of h-sheaves

$$0 \to K(B) \otimes \mathbb{Z}/\ell^n \to z_{equi}(B, 0)_h \otimes \mathbb{Z}/\ell^n \to I(B) \otimes \mathbb{Z}/\ell^n \to 0,$$

which induces a long exact sequence of hypercohomology

$$\cdots \to \mathbb{H}_h^{-1}(\mathrm{Spec}(k), C_*(z_{equi}(B, 0)_h \otimes \mathbb{Z}/\ell^n)) \to \mathbb{H}_h^{-1}(\mathrm{Spec}(k), C_*(I(B) \otimes \mathbb{Z}/\ell^n)) \to \mathbb{H}_h^0(\mathrm{Spec}(k), C_*(K(B) \otimes \mathbb{Z}/\ell^n)) \to \cdots.$$

Lemma 7.6. *Let X be a projective equi-dimensional variety and (B, Γ_B) be an equi-dimensional family of i-cycles on X parametrized by a projective variety B. Let T be a connected projective variety with a morphism $\Psi \colon T \to B$, let $\Gamma_T := \Psi^*\Gamma_B$, and assume that (T, Γ_T) is constant modulo ℓ^n. Then, for any $t_0, t_1 \in T$, we have*

$$[\Psi(t_0) - \Psi(t_1)] = 0 \in \mathbb{H}_h^0(\mathrm{Spec}(k), C_*(K(B) \otimes \mathbb{Z}/\ell^n)).$$

Proof. Fix $t_0 \in T$ and let

$$\theta \colon z_{equi}(T, 0)(\mathrm{Spec}(k)) \to z_{equi}(B, 0)(\mathrm{Spec}(k)),\ \alpha \mapsto \Psi_*(\alpha - \deg(\alpha)t_0).$$

Using that any dominant morphism to Δ^1 is flat, this extends to a map of complexes

$$\theta \colon \sigma_{\le 1}C_*(z_{equi}(T, 0))(\mathrm{Spec}(k)) \to \sigma_{\le 1}C_*(z_{equi}(B, 0))(\mathrm{Spec}(k)).$$

Since (T, Γ_T) is constant modulo ℓ^n by assumption, the composition

$$\sigma_{\le 1}C_*(z_{equi}(T, 0) \otimes \mathbb{Z}/\ell^n)(\mathrm{Spec}(k)) \xrightarrow{\theta} \sigma_{\le 1}C_*(z_{equi}(B, 0) \otimes \mathbb{Z}/\ell^n)(\mathrm{Spec}(k)) \xrightarrow{(\Gamma_W)_*} \sigma_{\le 1}C_*(z_{equi}(X, i) \otimes \mathbb{Z}/\ell^n)(\mathrm{Spec}(k))$$

is the zero map. Let $K(B)_{ps}$ be the kernel of the map $(\Gamma_B)_* \colon z_{equi}(B, 0) \to z_{equi}(X, i)$ of presheaves. Using that $z_{equi}(B, 0)$ and $z_{equi}(X, i)$ are torsion-free, the map $K(B)_{ps} \otimes \mathbb{Z}/\ell^n \to z_{equi}(B, 0) \otimes \mathbb{Z}/\ell^n$ of presheaves is injective, hence the map $\sigma_{\leq 1}C_*(K(B)_{ps}\otimes\mathbb{Z}/\ell^n)\to\sigma_{\leq 1}C(z_{equi}(B,0)\otimes\mathbb{Z}/\ell^n)$ of complexes of presheaves is injective. We then have an induced map

$$\theta\colon \sigma_{\leq 1}C(z_{equi}(T,0)\otimes\mathbb{Z}/\ell^n)(\mathrm{Spec}(k))\to\sigma_{\leq 1}C_*(K(B)_{ps}\otimes\mathbb{Z}/\ell^n)(\mathrm{Spec}(k)),$$

which yields

$$\begin{aligned}&CH_0(T)/\ell^n = H^0(\sigma_{\leq 1}C_*(z_{equi}(T,0)\otimes\mathbb{Z}/\ell^n)(\mathrm{Spec}(k)))\\ &\xrightarrow{\theta} H^0(\sigma_{\leq 1}C_*(K(B)_{ps}\otimes\mathbb{Z}/\ell^n)(\mathrm{Spec}(k))) = H^0(C_*(K(B)_{ps}\otimes\mathbb{Z}/\ell^n))(\mathrm{Spec}(k))\\ &\to H^0(C_*(K(B)\otimes\mathbb{Z}/\ell^n))(\mathrm{Spec}(k)) = \mathbb{H}^0_h(\mathrm{Spec}(k),C_*(K(B)\otimes\mathbb{Z}/\ell^n)),\end{aligned}$$

which maps $\alpha\in CH_0(T)/\ell^n$ to $[\Psi_*(\alpha-\deg(\alpha)t_0)]\in\mathbb{H}^0_h(\mathrm{Spec}(k),C_*(K(B)\otimes\mathbb{Z}/\ell^n))$. Since the group of 0-cycles of degree 0 on T modulo rational equivalence is divisible, the degree map induces an isomorphism $CH_0(T)/\ell^n\xrightarrow{\sim}\mathbb{Z}/\ell^n$. Hence, for any $t\in T$, we have

$$[\Psi(t)-\Psi(t_0)]=[\Psi(t_0)-\Psi(t_0)]=0\in\mathbb{H}^0_h(\mathrm{Spec}(k),C_*(K(B)\otimes\mathbb{Z}/\ell^n)).$$

This finishes the proof. □

We now prove Theorem 7.2.

Proof of Theorem 7.2. Let $\alpha\in CH_1(X,1,\mathbb{Z}/\ell^n)$ be in the kernel of $CH_1(X,1,\mathbb{Z}/\ell^n)\to A_1(X)[\ell^n]$. We aim to show that there exists an equi-dimensional family (W,Γ_W) of 1-cycles on X such that W is smooth projective and α lies in the image of $(\Gamma_W)_*\colon CH_0(B,1,\mathbb{Z}/\ell^n)\to CH_1(X,1,\mathbb{Z}/\ell^n)$. The proof is inspired by that of Theorem 4.3, but it needs some adjustments because we work with $\mathbb{Z}/\ell^n$-coefficients instead of $\mathbb{Z}$-coefficients.

By Lemma 7.5, α is represented by some equi-dimensional family $(\Delta^1,\Gamma_{\Delta^1})$ of 1-cycles on X. Writing $\Gamma_{\Delta^1}=\sum m_j\Gamma_j$, each projection $\Gamma_j\to\Delta^1$ is flat, hence it induces a morphism $\varphi_j\colon\Delta^1\to\mathrm{Hilb}(X)$. Let B be the normalization of the product of the irreducible components H_j of $\mathrm{Hilb}(X)$ which contain the images of φ_j, and let $\Gamma_B:=\sum m_j pr_j^*\varphi_j^*\,\mathrm{Univ}(X)$, where $\mathrm{Univ}(X)$ is the universal subscheme over $\mathrm{Hilb}(X)$ and $pr_j\colon B\to H_j$ are the projections. Then (B,Γ_B) is an equi-dimensional family of 1-cycles on X parametrized by the normal and projective variety B, and letting $\psi:=\prod\varphi_j\colon\Delta^1\to B$, we get $(\Delta^1,\psi^*\Gamma_B)=(\Delta^1,\Gamma_{\Delta^1})$. Let $b_0:=\psi(0)$, $b_1:=\psi(1)\in B$. We regard ψ as a path connecting b_0 and b_1.

By the choice of α, there exists a 1-cycle γ on X such that $(\Gamma_B)_{b_0}-(\Gamma_B)_{b_1}=\ell^n\gamma$ and γ is algebraically equivalent to 0. Then there exists an equi-dimensional family (C,Γ_C) of 1-cycles on X parametrized by a smooth projective connected curve C with $c_0,c_1\in C$ such that $(\Gamma_C)_{c_0}=\gamma$, $(\Gamma_C)_{c_1}=0$. Let $\widetilde{B}:=B\times C$, $\Gamma_{\widetilde{B}}:=pr_B^*\Gamma_B+\ell^n pr_C^*\Gamma_C$, and

$$\widetilde{b}_0:=(b_0,c_1),\qquad \widetilde{b}_1:=(b_1,c_1),\qquad \widetilde{b}_2:=(b_1,c_0)\in\widetilde{B}.$$

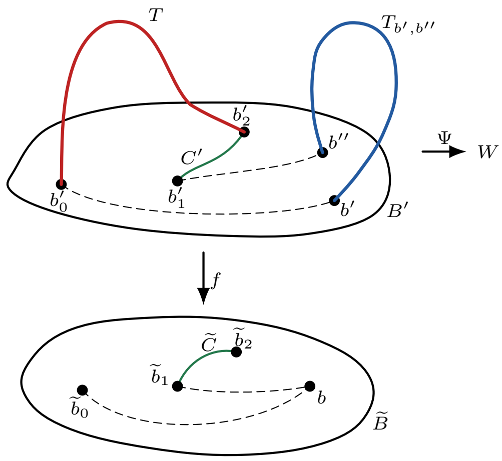
Then

$$(\Gamma_{\widetilde{B}})_{\widetilde{b}_0}=(\Gamma_B)_{b_0},\qquad (\Gamma_{\widetilde{B}})_{\widetilde{b}_1}=(\Gamma_B)_{b_1},\qquad (\Gamma_{\widetilde{B}})_{\widetilde{b}_2}=(\Gamma_B)_{b_1}+\ell^n\gamma=(\Gamma_B)_{b_0}$$

so that $(\Gamma_{\widetilde{B}})_{\widetilde{b}_0}=(\Gamma_{\widetilde{B}})_{\widetilde{b}_2}$. Let $\widetilde{\psi}\colon\Delta^1\xrightarrow{\psi}B\cong B\times\{c_1\}\hookrightarrow\widetilde{B}$. Then $\widetilde{\psi}$ connects $\widetilde{b}_0$ and $\widetilde{b}_1$, and $(\Delta^1,\widetilde{\psi}^*\Gamma_{\widetilde{B}})=(\Delta^1,\psi^*\Gamma_B)=(\Delta^1,\Gamma_{\Delta^1})$ represents α. In addition, let $\widetilde{C}:=\{b_1\}\times C$ and $\Gamma_{\widetilde{C}}:=\Gamma_{\widetilde{B}}|_{\widetilde{C}}$. Then $\widetilde{C}$ connects $\widetilde{b}_1$ and $\widetilde{b}_2$, and $(\widetilde{C},\Gamma_{\widetilde{C}})$ is constant modulo ℓ^n.

We apply Theorem 5.2 and get $(B',\Gamma_{B'})$, $b_0',b_2'\in B'$, $f\colon(B',\Gamma_{B'})\to(\widetilde{B},\Gamma_{\widetilde{B}})$ such that $f(b_i')=b_i$ $(i=0,2)$, (T,Γ_T), $(T_{b',b''},\Gamma_{b',b''})$ $(b',b''\in f^{-1}(b))$, and $\Psi\colon(B'\Gamma_{B'})\sqcup(T,\Gamma_T)\sqcup(\sqcup_{b',b''\in f^{-1}(b)}(T_{b',b''},\Gamma_{b',b''}))\to(W,\Gamma_W)$ as in the theorem. In addition, we choose a connected projective curve $C'\subset f^{-1}(\widetilde{C})$ such that C' maps onto $\widetilde{C}$ and $b_1'\in C'$, as well as a lift $b_2'\in\widetilde{C}$ of b_2, and let $\Gamma_{C'}$ be the pull-back of Γ_C along $C'\to C$. We aim to lift a path between $\widetilde{b}_0$ and $\widetilde{b}_1$ along f, and close it up using the constant families (T,Γ_T), $(T_{b',b''},\Gamma_{b',b''})$

and the constant-modulo-ℓ^n family $(C', \Gamma_{C'})$.



A formal proof is as follows. The short exact sequences of h-sheaves

$$0 \to K(\widetilde{B}) \to z_{equi}(\widetilde{B}, 0)_h \to I(\widetilde{B}) \to 0, \quad 0 \to K(B') \to z_{equi}(B', 0)_h \to I(B') \to 0, \quad 0 \to K(W) \to z_{equi}(W, 0)_h \to I(W) \to 0$$

together with the fact that $(\Gamma_{B'})_* \colon z_{equi}(B', 0)_h \to z_{equi}(X, 1)_h$ factors as

$$z_{equi}(B', 0)_h \xrightarrow{f_*} z_{equi}(\widetilde{B}, 0)_h \xrightarrow{(\Gamma_{\widetilde{B}})_*} z_{equi}(X, 1)_h, \qquad z_{equi}(B', 0)_h \xrightarrow{\Psi_*} z_{equi}(W, 0)_h \xrightarrow{(\Gamma_W)_*} z_{equi}(X, 1)_h$$

yield the following commutative diagram of hypercohomology

$$\begin{array}{ccccc}
\mathbb{H}_h^{-1}(\mathrm{Spec}(k), C_*(z_{equi}(W,0)_h \otimes \mathbb{Z}/\ell^n)) & \longrightarrow & \mathbb{H}_h^{-1}(\mathrm{Spec}(k), C_*(I(W) \otimes \mathbb{Z}/\ell^n)) & \longrightarrow & \mathbb{H}_h^{0}(\mathrm{Spec}(k), C_*(K(W) \otimes \mathbb{Z}/\ell^n)) \\
\uparrow \Psi_* & & \uparrow \Psi_* & & \uparrow \Psi_* \\
\mathbb{H}_h^{-1}(\mathrm{Spec}(k), C_*(z_{equi}(B',0) \otimes \mathbb{Z}/\ell^n)) & \longrightarrow & \mathbb{H}_h^{-1}(\mathrm{Spec}(k), C_*(I(B') \otimes \mathbb{Z}/\ell^n)) & \longrightarrow & \mathbb{H}_h^{0}(\mathrm{Spec}(k), C_*(K(B') \otimes \mathbb{Z}/\ell^n)) \\
\downarrow f_* & & \wr\downarrow f_* & & \downarrow f_* \\
\mathbb{H}_h^{-1}(\mathrm{Spec}(k), C_*(z_{equi}(\widetilde{B},0) \otimes \mathbb{Z}/\ell^n)) & \longrightarrow & \mathbb{H}_h^{-1}(\mathrm{Spec}(k), C_*(I(\widetilde{B}) \otimes \mathbb{Z}/\ell^n)) & \longrightarrow & \mathbb{H}_h^{0}(\mathrm{Spec}(k), C_*(K(\widetilde{B}) \otimes \mathbb{Z}/\ell^n)),
\end{array}$$

where $f_* \colon \mathbb{H}_h^{-1}(\mathrm{Spec}(k), C_*(I(B') \otimes \mathbb{Z}/\ell^n)) \xrightarrow{\sim} \mathbb{H}_h^{-1}(\mathrm{Spec}(k), C_*(I(\widetilde{B}) \otimes \mathbb{Z}/\ell^n))$ because $f_* \colon z_{equi}(B', 0)_h \to z_{equi}(\widetilde{B}, 0)_h$ is surjective by Lemma 7.4 and thus $I(B') \xrightarrow{\sim} I(\widetilde{B})$. Let $\alpha' \in \mathbb{H}_h^{-1}(\mathrm{Spec}(k), C_*(I(B') \otimes \mathbb{Z}/\ell^n))$ be a unique lift of $\alpha \in \mathbb{H}_h^{-1}(\mathrm{Spec}(k), C_*(I(\widetilde{B}) \otimes \mathbb{Z}/\ell^n))$.

We first show that $\Psi_*\alpha' \in \mathbb{H}_h^{-1}(\mathrm{Spec}(k), C_*(I(W) \otimes \mathbb{Z}/\ell^n))$ lifts to $\mathbb{H}_h^{-1}(\mathrm{Spec}(k), C_*(z_{equi}(W,0)_h \otimes \mathbb{Z}/\ell^n))$, which will imply that for any lift $\alpha_W \in \mathbb{H}_h^{-1}(\mathrm{Spec}(k), C_*(z_{equi}(W,0)_h \otimes \mathbb{Z}/\ell^n))$ of $\Psi_*\alpha'$, we have $(\Gamma_W)_*\alpha_W = \alpha$. To prove the assertion, it is enough to show that the image of α' in $\mathbb{H}_h^0(\mathrm{Spec}(k), C_*(K(W) \otimes \mathbb{Z}/\ell^n))$ vanishes, and using the short exact sequence $0 \to \mathrm{Ker}(f_*) \to K(B') \to K(\widetilde{B}) \to 0$ of h-sheaves, we see that the image of α' in $\mathbb{H}_h^0(\mathrm{Spec}(k), C_*(K(B') \otimes \mathbb{Z}/\ell^n))$ is represented by $[b'_0 - b'_1]$ modulo $\mathbb{H}_h^0(\mathrm{Spec}(k), C_*(\mathrm{Ker}(f_*) \otimes \mathbb{Z}/\ell^n))$. Moreover, as before, one can show that $\mathbb{H}_h^0(\mathrm{Spec}(k), C_*(\mathrm{Ker}(f_*) \otimes \mathbb{Z}/\ell^n))$ is generated by $[b' - b'']$ ($b', b'' \in f^{-1}(b)$); see [24, Lemma 5.9]. Finally, Lemma 7.6 applied to $T, T_{b',b''}, C'$ yields

$$[\Psi(b'_1) - \Psi(b'_0)] = [\Psi(b'_1) - \Psi(b'_2)] + [\Psi(b'_2) - \Psi(b'_0)] = 0, \qquad [\Psi(b'') - \Psi(b')] = 0$$

in $\mathbb{H}_h^0(\mathrm{Spec}(k), C_*(K(W) \otimes \mathbb{Z}/\ell^n))$. The assertion follows.

By the previous paragraph, $\alpha \in CH_1(X, 1, \mathbb{Z}/\ell^n)$ is in the image of

$$(\Gamma_W)_* \colon CH_0(W, 1, \mathbb{Z}/\ell^n) \to CH_1(X, 1, \mathbb{Z}/\ell^n).$$

By [12, Theorem 2.1], there exists a smooth projective connected variety W' and a generically finite morphism $g \colon W' \to W$ of degree prime to ℓ. Then $g_* \colon CH_0(W', 1, \mathbb{Z}/\ell^n) \to CH_0(W, 1, \mathbb{Z}/\ell^n)$ is surjective, and letting $\Gamma_{W'} := g^*\Gamma_W$,

$$(\Gamma_{W'})_* = (\Gamma_W)_* g_* \colon CH_0(W', 1, \mathbb{Z}/\ell^n) \to CH_1(X, 1, \mathbb{Z}/\ell^n)$$

has the same image as that of $(\Gamma_W)_*$. Hence $\alpha \in CH_1(X,1,\mathbb{Z}/\ell^n)$ is in the image of $(\Gamma_{W'})_*$, and this finishes the proof. □

7.3. **Proof of Theorem 4.3.** Let X be an equi-dimensional quasi-projective variety. There is a cycle map

$$CH_i(X,j,\mathbb{Z}/\ell^n) \to H_{2i+j}(X,\mathbb{Z}/\ell^n).$$

We will be particularly interested in the case $j=1$. By the Beilinson–Lichtenbaum conjecture settled by Voevodsky, one gets:

Lemma 7.7. *Let X be a smooth projective connected variety. We have*

$$N_{i+1}H_{2i+1}(X,\mathbb{Z}/\ell^n) = \mathrm{Im}(CH_i(X,1,\mathbb{Z}/\ell^n) \to H_{2i+1}(X,\mathbb{Z}/\ell^n)).$$

Lemma 7.7 may be compared with Lemma 6.3.

Proof of Theorem 4.3. The maps

$$\widetilde{N}_1H_3(X,\mathbb{Z}_\ell)/\ell^n \to N_1H_3(X,\mathbb{Z}_\ell)/\ell^n \to N_1H_3(X,\mathbb{Z}/\ell^n)\cap H_3(X,\mathbb{Z}_\ell)/\ell^n \to H_3(X,\mathbb{Z}_\ell)/\ell^n$$

induces the inclusions

$$\widetilde{N}_1H_3(X,\mathbb{Z}_\ell) \hookrightarrow N_1H_3(X,\mathbb{Z}_\ell) \hookrightarrow \varprojlim N_1H_3(X,\mathbb{Z}/\ell^n)\cap H_3(X,\mathbb{Z}_\ell)/\ell^n \hookrightarrow H_3(X,\mathbb{Z}_\ell).$$

Hence it suffices for us to show that

$$\widetilde{N}_1H_3(X,\mathbb{Z}_\ell) = \varprojlim N_1H_3(X,\mathbb{Z}/\ell^n)\cap H_3(X,\mathbb{Z}_\ell)/\ell^n.$$

We will establish this by showing that the natural maps

$$\varphi_n\colon \widetilde{N}_1H_3(X,\mathbb{Z}_\ell) \to N_1H_3(X,\mathbb{Z}/\ell^n)\cap H_3(X,\mathbb{Z}_\ell)/\ell^n$$

satisfy $\varprojlim \mathrm{Coker}(\varphi_n) = 0$.

By Theorem 7.2, the cycle maps for higher Chow groups induce a commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & \bigoplus_{(B,\Gamma)} CH_0(B,1,\mathbb{Z}/\ell^n) & \xrightarrow{(\Gamma_*)} & CH_1(X,1,\mathbb{Z}/\ell^n) & \longrightarrow & A_1(X)[\ell^n] & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H_3(X,\mathbb{Z}_\ell)/\ell^n & \longrightarrow & H_3(X,\mathbb{Z}/\ell^n) & \longrightarrow & H_2(X,\mathbb{Z}_\ell)[\ell^n] & \longrightarrow & 0, \end{array}$$

where (B,Γ) runs over smooth projective varieties B and $\Gamma \in CH_{\dim B+1}(B\times X)$, and this induces another commutative diagram with exact rows

$$\begin{array}{ccccccc} & & \bigoplus_{(B,\Gamma)} CH_0(B,1,\mathbb{Z}/\ell^n) & \xrightarrow{(\Gamma_*)} & CH_1(X,1,\mathbb{Z}/\ell^n) & \longrightarrow & A_1(X)[\ell^n] \\ & & \downarrow & & \twoheaddownarrow & & \downarrow \\ 0 & \longrightarrow & N_1H_3(X,\mathbb{Z}_\ell/\ell^n)\cap H_3(X,\mathbb{Z}_\ell)/\ell^n & \longrightarrow & N_1H_3(X,\mathbb{Z}/\ell^n) & \longrightarrow & H_2(X,\mathbb{Z}_\ell)[\ell^n], \end{array} \tag{7.1}$$

where (B,Γ) runs over all B,Γ as above. By Lemma 7.7, the middle vertical arrow is surjective. Since $CH_0(B,1,\mathbb{Z}/\ell^n) \xrightarrow{\sim} H_1(B,\mathbb{Z}/\ell^n) = H_1(B,\mathbb{Z}_\ell)/\ell^n$, by Lemma 4.2 togethr with the compatibility of the cycle maps and the action of correspondences, the image of the left vertical arrow of (7.1) agrees with the image of φ_n. Applying the snake lemma to (7.1) then shows that the cokernel of φ_n is a subquotient of $A_1(X)[\ell^n]$.

Finally, by Bloch–Srinivas [4, Theorem 2 (i)] and the finiteness of the torsion subgroup of $H_2(X,\mathbb{Z}_\ell)$, there exists $N>0$ such that $A_1(X)\{\ell\} = A_1(X)[\ell^N]$. This shows $\varprojlim A_1(X)[\ell^n] = 0$, and since the Mittag–Leffler condition holds for any projective system which injects to the projective system $\cdots \xrightarrow{\ell} A_1(X)[\ell^{n+1}] \xrightarrow{\ell} A_1(X)[\ell^n] \xrightarrow{\ell} \cdots$, we have $\varprojlim \mathrm{Coker}(\varphi_n) = 0$, as desired. The proof is complete. □

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