An $\mathcal{O}$-acyclic variety of even index

joint with John Christian Ottem

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X : a smooth projective variety over $\mathbb{C}$
B: a smooth projective curve
$X \to B$: a fibration

Graber-Harris-Starr: $\exists$ section if X_η is rationally connected.

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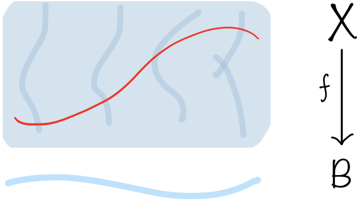


Graber-Harris-Starr: $\exists$ section if X_η is rationally connected.

Question (Serre, 1958)

Does $X \to B$ admit a section if X_η is $\mathcal{O}$-acyclic, that is, $H^i(X_\eta, \mathcal{O}_{X_\eta}) = 0$ for all $i > 0$?

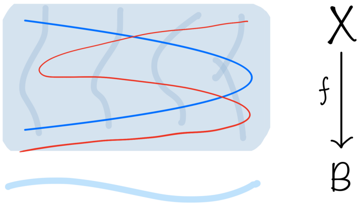
Graber-Harris-Mazur-Starr constructed a **counterexample**:
$\exists$ an Enriques surface fibration $X \to B$ without section.

Lafon, Starr: more explicit constructions

Question

Does $X \to B$ have index $I(X/B) = 1$ if X_η is $\mathcal{O}$-acyclic?

$$I(X/B) = \gcd\{\deg(M/B) \mid M \text{ is a multi-section of } X \to B\}$$



There is no local obstruciton: any $X \to B$ with X_η $\mathcal{O}$-acyclic has no multiple fiber ($\Leftrightarrow I(X/B) = 1$ everywhere locally).

Esnault and Colliot-Thélène–Voisin expected a negative answer.

Theorem (Ottem-S.)

There exists an Enriques surface fibration $X \to \mathbb{P}^1$ of even index.

One can find $X \to \mathbb{P}^1$ as in the theorem defined over $\mathbb{Q}$.

The index question has a positive answer over $\overline{\mathbb{F}}_p$ if we further assume the Tate conjecture and $b_2(X) = \rho(X)$.

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I. Let $X \to \mathbb{P}^1$ be the Enriques surface fibration of the main theorem. Then $Y = X_\eta$ is an Enriques surface over $F = \mathbb{C}(\mathbb{P}^1)$.

Y does not admit a 0-cycle of degree 1, while Y_{F_p} does for any $p \in \mathbb{P}^1$, where $F_p \cong \mathbb{C}((t))$ is the completion of F at p.
$\Rightarrow$ The Hasse principle fails for 0-cycles of degree 1 on Y.

Question (Colliot-Thélène)

Can the failure of the Hasse principle on Y be accounted for by the reciprocity obstruction?

No. In fact:

Theorem (Wittenberg)

For any smooth projective $\mathcal{O}$-acyclic variety Y over the function field $F = \mathbb{C}(B)$ of a complex curve B, there is no reciprocity obstruction.

II. For X smooth projective over $\mathbb{C}$, we have:

$$H^{2p}_{\mathrm{alg}}(X,\mathbb{Z}) := \mathrm{Im}\left(\mathrm{cl}^p\colon CH^p(X) \to H^{2p}(X,\mathbb{Z})\right)$$
$$\subseteq Hdg^{2p}(X,\mathbb{Z}) := H^{2p}(X,\mathbb{Z}) \cap H^{p,p}(X).$$

Integral Hodge Conjecture (IHC): $H^{2p}_{\mathrm{alg}}(X,\mathbb{Z}) = Hdg^{2p}(X,\mathbb{Z})$.

IHC holds for $p = 0, \dim X$ (trivial), $p = 1$.

There are counterexamples for $2 \leq p \leq \dim X - 1$:
Atiyah-Hirzebruch, Kollár, $\cdots$

Theorem (Colliot-Thélène–Voisin)

If $f: X \to B$ is a fibration with X_η $\mathcal{O}$-acyclic, then

$$f_*: H_2(X,\mathbb{Z}) = Hdg_2(X,\mathbb{Z}) \twoheadrightarrow H_2(B,\mathbb{Z}) = \mathbb{Z}.$$

Hence $\exists \beta \in Hdg_2(X,\mathbb{Z}) \mapsto 1 \in H_2(B,\mathbb{Z})$.

Corollary

If $X \to \mathbb{P}^1$ is the Enriques surface fibration of the main theorem, then

$$H^4_{\mathrm{alg}}(X,\mathbb{Z})/\operatorname{tors} \subsetneq Hdg^4(X,\mathbb{Z})/\operatorname{tors}.$$

β is not algebraic; 4β is algebraic on our example.

III. Consider the Abel-Jacobi map for smooth projective V:

$$AJ^p \colon CH^p_{\mathrm{hom}}(V) \to J^{2p-1}(V) = J(H^{2p-1}(V,\mathbb{Z})).$$

Restrict AJ^p to $CH^p_{\mathrm{alg}}(V)$:

$$\psi^p \colon CH^p_{\mathrm{alg}}(V) \to J^{2p-1}_a(V) := AJ^p(CH^p_{\mathrm{alg}}(V)).$$

Then $J^{2p-1}_a(V)$ is an abelian variety and ψ^p is regular:
for any smooth projective S with $s_0 \in S$ and any $\Gamma \in CH^p(S \times V)$,

$$S \to CH^p_{\mathrm{alg}}(V) \xrightarrow{\psi^p} J^{2p-1}_a(V),\ s \mapsto \psi^p(\Gamma_s - \Gamma_{s_0})$$

is a morphism of algebraic varieties (Griffiths, Lieberman).

Question (Murre, 1985)

Is ψ^p universal among all regular homomorphisms?

$$\begin{array}{ccc} & & J_a^{2p-1}(V)\ . \\ & \overset{\psi^p}{\nearrow} & \vdots\ \exists? \\ CH^p_{\mathrm{alg}}(V) & \xrightarrow[\phi]{\qquad} & A \end{array}$$

Yes for $p = 1, \dim V$ by the theory of Pic and Alb.
Yes for $p = 2$ as proved by Murre using the Merkurjev-Suslin theorem.

Walker: the Abel-Jacobi map ψ^p factors as

$$\begin{array}{ccc} & & J_W^{2p-1}(V), \\ & \overset{\psi^p_W}{\nearrow} & \downarrow \pi^p \\ CH^p_{\mathrm{alg}}(V) & \xrightarrow[\psi^p]{} & J_a^{2p-1}(V) \end{array}$$

where

- ψ^p_W is a surjective regular homomorphism;
- $J_W^{2p-1}(V) = J(N^{p-1}H^{2p-1}(V,\mathbb{Z}))$;
- π^p is a natural isogeny of abelian varieties.

Therefore

$$\psi^p \text{ is universal} \Rightarrow \mathrm{Ker}(\pi^p) = 0,$$

or equivalently, the sublattice

$$N^{p-1}H^{2p-1}(V,\mathbb{Z})/\operatorname{tors} \subset H^{2p-1}(V,\mathbb{Z})/\operatorname{tors}$$

is saturated.

Theorem (S.)

Let X be smooth projective such that

1. $H^4_{\text{alg}}(X,\mathbb{Z})/\text{tors} \subsetneq Hdg^4(X,\mathbb{Z})/\text{tors}$;
2. $CH_0(X)$ *is supported on a surface.*

Then there exists an elliptic curve E such that the sublattice

$$N^2H^5(X\times E,\mathbb{Z})/\text{tors} \subset H^5(X\times E,\mathbb{Z})/\text{tors}$$

is **NOT** *saturated. Consequently, the Abel-Jacobi map*

$$\psi^3\colon CH^3_{\text{alg}}(X\times E)\to J^5_a(X\times E)$$

is **NOT** *unviersal.*

The theorem can be applied to X of the main theorem.
In fact, $CH_0(X)=\mathbb{Z}$.
This settles Murre's question.

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Let $S = (1)^3 \subset \mathbb{P}_{\mathbb{P}^2 \times \mathbb{P}^2}(\mathcal{O}(2,0) \oplus \mathcal{O}(0,2))$.

Lemma

If S is general, S is an Enriques surface.

Proof:

$$\begin{array}{rcl} \mathbb{P}_A = \mathbb{P}_{\mathbb{P}^2\times\mathbb{P}^2}(\mathcal{O}(2,0)\oplus\mathcal{O}(0,2)) & \supseteq & E_1, E_2 \\ \mathbb{P}_B = \mathbb{P}_{\mathbb{P}^2\times\mathbb{P}^2}(\mathcal{O}(1,0)\oplus\mathcal{O}(0,1)) & \supseteq & F_1, F_2 \\ \mathbb{P}^5 = \mathbb{P}(H^0(\mathcal{O}(1,0)\oplus\mathcal{O}(0,1))) & \supseteq & P_1, P_2. \end{array}$$

$$\begin{array}{ccccc} & & \mathbb{P}_B & & \\ & \swarrow^{bl_{P_1 \amalg P_2}} & \cup\!\mid & \searrow^{2:1} & \\ \mathbb{P}^5 & & (2)^3 & & \mathbb{P}_A \\ \cup\!\mid & \swarrow^{\cong} & & \searrow^{\text{étale double}} & \cup\!\mid \\ (2)^3 & & & & S = (1)^3 \end{array}.$$

It is enough to observe that $(2)^3 \subset \mathbb{P}^5$ is a K3 surface.

Q.E.D.

Let $X = (2,1)^3 \subset \mathbb{P}^1 \times \mathbb{P}_A$.
(Convention: $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}_A}(a,b) = pr_1^* \mathcal{O}_{\mathbb{P}^1}(a) \otimes pr_2^* \mathcal{O}_{\mathbb{P}_A}(b)$).

If X is general, then $pr_1 \colon X \to \mathbb{P}^1$ is an Enriques surface fibration.

Lemma

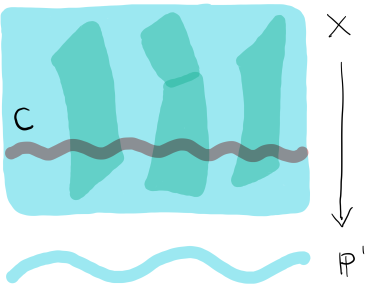
1. $\kappa(X) = 1$.
2. *X is simply connected and $H^i(X, \mathbb{Z})$ are torsion-free for all i.*
3. *Hodge diamond*

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 50 & & 0 & \\ 0 & & 99 & & 99 & & 0 \end{array}$$

4. *$CH_0(X) = \mathbb{Z}$ (as expected by the Bloch conjecture).*

Proof that $CH_0(X) = \mathbb{Z}$:

Let $C \subset X$ be a complete intersection curve.



Bloch-Kas-Lieberman: $CH_0(S) = \mathbb{Z}$ for any Enriques surfaces S.
$\Rightarrow CH_0(X)$ is supported on C.
$\Rightarrow CH_0(X)_{\mathrm{deg}=0} \cong \mathrm{Alb}(X) = 0$.

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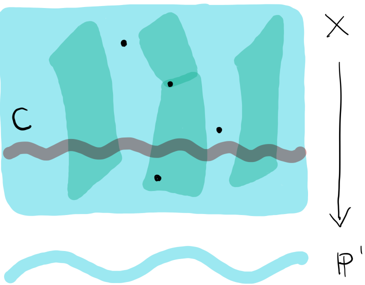

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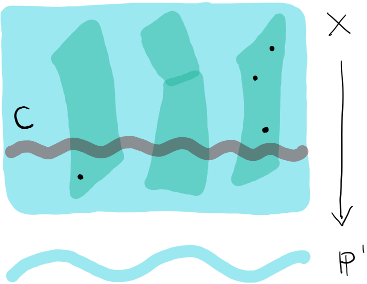

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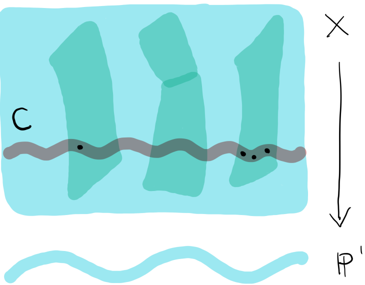

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Q.E.D.

Geometry of X:

$$
\begin{array}{ccccc}
 & & \mathbb{P}^1 \times \mathbb{P}_B & & \\
 & \swarrow^{bl_{\mathbb{P}^1 \times P_i}} & \rotatebox{90}{$\supseteq$} & \searrow^{2:1} & \\
\mathbb{P}^1 \times \mathbb{P}^5 & & (2,2)^3 & & \mathbb{P}^1 \times \mathbb{P}_A \\
\rotatebox{90}{$\supseteq$} & \swarrow_{bl_{Q_{i,j}}} & \rotatebox{90}{$\supseteq$} & \searrow^{2:1} & \rotatebox{90}{$\supseteq$} \\
(2,2)^3 & & F_{i,j} & & X = (2,1)^3 \\
\rotatebox{90}{$\in$} & & & & \rotatebox{90}{$\supseteq$} \\
Q_{i,j} & & & & E_{i,j}
\end{array}
$$

$(i = 1, 2,\ j = 1, \cdots, 24)$

The 24 planes $X \cap (\mathbb{P}^1 \times E_1) = E_{1,1} \cup \cdots \cup E_{1,24}$ will be important.

Theorem

If X is very general, any multi-section of $pr_1 : X \to \mathbb{P}^1$ has even degree over $\mathbb{P}^1$.

Proof:

We aim to show a **key congruence**: for any multi-section M and for any 12-tuple $1 \leq j_1 < \cdots < j_{12} \leq 24$, we have

$$\deg(M/\mathbb{P}^1) \equiv \sum_{k=1}^{12} M \cdot E_{1,j_k} \mod 2. \tag{1}$$

(1) $\Rightarrow$ Theorem: $M \cdot E_{1,1} \equiv \cdots \equiv M \cdot E_{1,24}$, so $\deg(M/\mathbb{P}^1)$ is even.

The proof is a combination of monodromy and specialization arguments.

Step 1: A monodromy argument reduces the theorem to showing the congruence (1) for a **single** 12-tuple $1 \leq j_1 < \cdots < j_{12} \leq 24$.

Step 2: A specialization argument. Use specializations twice.

Step 1:

Let $\mathcal{X} \to U$ be the universal family of X.

There is a natural action of $\pi_1(U)$ on the set of 24 planes $E_{1,1}, \cdots, E_{1,24}$ by permutations.

Lemma

The monorodomy representation $\rho\colon \pi_1(U) \to S_{24}$ is surjectve.

Assume that the congruence (1) holds for a 12-tuple $1 \le j_1 < \cdots < j_{12} \le 24$. If $\sigma \in S_{24}$ and $g \in \pi_1(U)$ is a lift of σ,

$$\deg(M/\mathbb{P}^1) \equiv \deg(g^*(M)/\mathbb{P}^1) \equiv \sum_{k=1}^{12} M \cdot E_{1,\sigma^{-1}(j_k)} \mod 2$$

for any multi-section M of $pr_1\colon X \to \mathbb{P}^1$.

Step 2:

Let $Y = (1,1)^3 \subset \mathbb{P}^1 \times \mathbb{P}_A$.

If Y is general, then $pr_1 \colon Y \to \mathbb{P}^1$ is an Enriques surface fibration.

Lemma

1. $\kappa(Y) = 1$.
2. *Y is simply connected and $H^i(Y,\mathbb{Z})$ are torsion-free for all i.*
3. *Hodge diamond*

$$\begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 26 & & 0 & \\ 0 & & 45 & & 45 & & 0 \end{matrix}$$

4. *$CH_0(Y) = \mathbb{Z}$ (as expected by the Bloch conjecture).*

Geometry of Y:

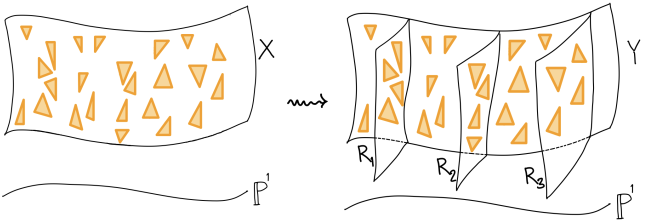
$$\begin{array}{ccccc} & & \mathbb{P}^1 \times \mathbb{P}_B & & \\ & \swarrow^{bl_{\mathbb{P}^1 \times P_i}} & \cup\!| & \searrow^{2:1} & \\ \mathbb{P}^1 \times \mathbb{P}^5 & & (1,2)^3 & & \mathbb{P}^1 \times \mathbb{P}_A \\ \cup\!| & \swarrow_{bl_{Q_{i,j}}} & \cup\!| & \searrow^{2:1} & \cup\!| \\ (1,2)^3 & & F'_{i,j} & & Y = (1,1)^3 \\ \Cup & & & & \cup\!| \\ Q'_{i,j} & & & & E'_{i,j} \end{array}$$

$(i = 1, 2,\ j = 1, \cdots, 12)$.

The 12 planes $Y \cap (\mathbb{P}^1 \times E_1) = E'_{1,1} \cup \cdots \cup E'_{1,12}$ will be important.

First specialization:

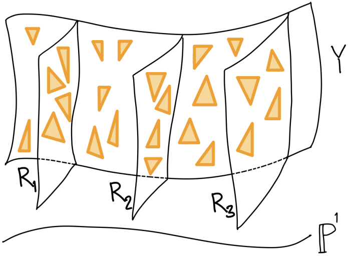
Specialize $X = (2,1)^3$ to the union of
$Y = (1,1)^3$ and $R_1, R_2, R_3 = (1,0) \cap (0,1)^2$ in $\mathbb{P}^1 \times \mathbb{P}_A$.



Under the chosen specialization,

$$\{E_{1,j}\}_{j=1}^{24} \mapsto \left\{E'_{1,j}\right\}_{j=1}^{12} \cup \left\{E_1^{(1)}, \cdots, E_4^{(1)}\right\} \cup \cdots \cup \left\{E_1^{(3)}, \cdots, E_4^{(3)}\right\}.$$

This determines a 12-tuple $1 \le j_1 < \cdots < j_{12} \le 24$ such that $E_{1,j_1}, \cdots, E_{1,j_{12}}$ specialize to $E'_{1,1}, \cdots, E'_{1,12}$.



We wanted to show: for any multi-section M of $pr_1 : X \to \mathbb{P}^1$, we have

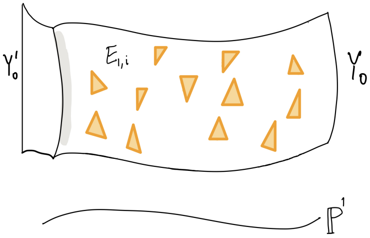
$$\deg(M/\mathbb{P}^1) \equiv \sum_{k=1}^{12} M \cdot E_{1,j_k} \mod 2.$$

Enough to show: for any multi-section M of $pr_1 : Y \to \mathbb{P}^1$, we have

$$\deg(M/\mathbb{P}^1) \equiv \sum_{j=1}^{12} M \cdot E'_{1,j} \mod 2.$$

Second specialization:

Specialize Y to a union $Y_0 \cup Y_0'$ such that $F - \sum_{j=1}^{12} E'_{1,j}$ is double on each component (but not on the union).



Q.E.D.

Thank you for the attention!